2011 秋 統計

(1) $\mathcal{H} = \epsilon(\vec{p}) + V(\vec{r})$

$$\epsilon(\vec{p}) = \epsilon(p) = \frac{p^2}{2m}$$

$$\langle \epsilon(p) \rangle = \frac{\iiint dp_x dp_y dp_z \frac{p^2}{2m} e^{-\beta\frac{p^2}{2m}}}{\iiint dp_x dp_y dp_z\, e^{-\beta\frac{p^2}{2m}}}$$

$$= \frac{\cancel{4\pi}\int_0^\infty dp \frac{p^4}{2m} e^{-\beta\frac{p^2}{2m}}}{\cancel{4\pi}\int_0^\infty dp\, p^2 e^{-\beta\frac{p^2}{2m}}} = \frac{1}{2m}\frac{\frac{3\sqrt{\pi}}{4\cdot 2}\left(\frac{\beta}{2m}\right)^{-\frac{5}{2}}}{\frac{\sqrt{\pi}}{2}\left(\frac{\beta}{2m}\right)^{-\frac{3}{2}}} = \frac{3}{4m}\left(\frac{\beta}{2m}\right)^{-1}$$

$$= \frac{3}{2\beta}$$

$$= \frac{1}{2m}\frac{\int p^2 e^{-\beta\frac{p^2}{2m}} dp}{\int e^{-\beta\frac{p^2}{2m}} dp} = \frac{1}{2m}\frac{\frac{\cancel{\sqrt{\pi}}}{2}\left(\frac{\beta}{2m}\right)^{-\frac{3}{2}}}{\cancel{\sqrt{\pi}}\left(\frac{\beta}{2m}\right)^{-\frac{1}{2}}}$$

$$= \frac{1}{4m}\left(\frac{\beta}{2m}\right)^{-1}$$

$$= \cancel{\frac{1}{2\beta}}$$

(2) $\epsilon(p) = cp$

$$\langle \epsilon(p) \rangle = \frac{4\pi\int_0^\infty cp^3 e^{-\beta cp} dp}{4\pi\int_0^\infty p^2 e^{-\beta cp} dp} = c\frac{\left[-p\frac{1}{\beta c}e^{-\beta cp}\right]_0^\infty + \int\frac{1}{c\beta}e^{-\beta cp}dp}{\int \cancel{e^{-\beta cp}}\, dp}$$

$$= c\frac{\int 3p^2\frac{1}{\beta c}e^{-\beta cp}dp}{\int p^2 e^{-\beta cp} dp} = \frac{3}{\beta} = 3k_BT$$

$$= \cancel{c^2\beta}$$

(3)

$$\langle G(\vec{r}) \rangle = \frac{\int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} dz\, G(\vec{r})\, e^{-\beta V(\vec{r})}}{\int_0^{L_x} dx \int dy \int dz\, e^{-\beta V(\vec{r})}}$$

$$\langle G(\vec{z}) \rangle = \frac{\int_0^{L_z} G(z)\, e^{-\beta V(z)}\, dz}{\int e^{-\beta V(z)}\, dz}$$

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(4) $V(z) = mgz$

$$\langle z \rangle = \frac{\int_0^{L_z} z e^{-\beta mgz} dz}{\int_0^{L_z} e^{-\beta mgz} dz}$$

$$= \frac{\left[-z\frac{1}{\beta mg}e^{-\beta mgz}\right]_0^{L_z} + \int \frac{1}{\beta mg} e^{-\beta mgz} dz}{\int_0^{L_z} e^{-\beta mgz} dz}$$

$$= \left\{ \frac{\left[-\frac{1}{\beta mg} e^{-\beta mgz}\right]_0^{L_z}}{\left[-\frac{z}{\beta mg} e^{-\beta mgz}\right]_0^{L_z}} \right\}^{-1} + \frac{1}{\beta mg}$$

$$= \frac{-\frac{L_z}{\beta mg} e^{-\beta mg L_z}}{-\frac{1}{\beta mg}(e^{-\beta mg L_z} - 1)} + \frac{1}{\beta mg} = \frac{L_z e^{-\beta mg L_z}}{e^{-\beta mg L_z} - 1} + \frac{1}{\beta mg}$$

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(5) $\langle z \rangle \simeq \lim_{L_z \to \infty} \langle z \rangle = \frac{1}{\beta mg}$

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