ラグランジアン $L(q,\dot{q},t)$

作用 $S=\int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$

$q(t_1)=q^{(1)},\quad q(t_2)=q^{(2)}$

$\delta S=0$ (最小作用の原理)

E-L 方程式

$\delta S = S'-S$

$$S'=\int_{t_1}^{t_2} L(q+\delta q,\dot{q}+\delta\dot{q},t)\,dt$$

$$q\to q+\delta q$$

$$\dot{q}\to\dot{q}+\underbrace{\frac{d}{dt}(\delta q)}_{\delta\dot{q}}$$

$$\delta S=\int_{t_1}^{t_2}\Big[\underbrace{L(q+\delta q,\dot{q}+\delta\dot{q},t)}_{\text{テイラー展開}}-L(q,\dot{q},t)\Big]dt$$

$\delta q(t_1)=\delta q(t_2)=0$ …条件.

$$L(q+\delta q,\dot{q}+\delta\dot{q},t)$$
$$=L(q,\dot{q},t)+\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial\dot{q}}\delta\dot{q}+\cdots$$

$$\delta S=\int_{t_1}^{t_2}\Big(\frac{\partial L}{\partial q}\delta q+\underbrace{\frac{\partial L}{\partial\dot{q}}\delta\dot{q}}_{\text{部分積分}}\Big)dt$$

$$\int_{t_1}^{t_2}\frac{\partial L}{\partial\dot{q}}\underbrace{\delta\dot{q}}_{=0}\,dt=\Big[\frac{\partial L}{\partial\dot{q}}\delta q\Big]_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}\delta q$$

$$\delta S=\int_{t_1}^{t_2}\Big(\frac{\partial L}{\partial q}-\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}\Big)\delta q\,dt$$

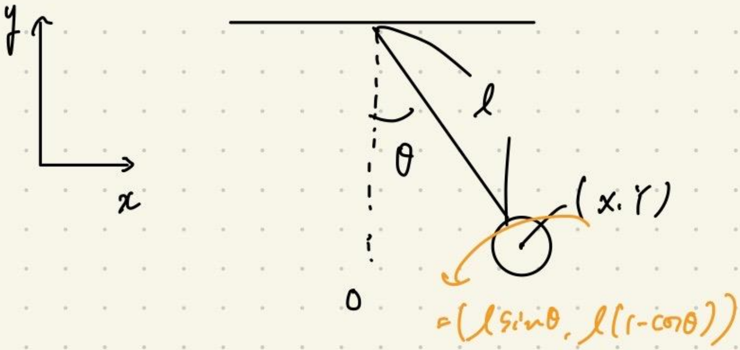
$\delta S=0$ より

$\dfrac{\partial L}{\partial q}-\dfrac{d}{dt}\dfrac{\partial L}{\partial\dot{q}}=0$: E-L方程式

↓

$$\frac{\partial L}{\partial q}=\frac{d}{dt}\frac{\partial L}{\partial\dot{q}}=\Big(\frac{d}{dt}\Big)\frac{\partial L}{\frac{d}{dt}\partial q}$$

→ こう覚える.



保存力下での質点系のラグランジアン

$L=\underbrace{K}_{\text{運動エネルギー}}-\underline{U}$ ポテンシャル

で与えられる.

(1) ラグランジアン

$$L=K-U$$
$$=\frac{1}{2}mv^2-mgh$$

$$v^2=|\mathbf{v}|^2=|\dot{\mathbf{r}}|^2$$

$$\mathbf{r}=(x,y)$$
$$=(\ell\sin\theta,\ \ell(1-\cos\theta))$$

$$\dot{\mathbf{r}}=(\ell\dot{\theta}\cos\theta,\ \ell\dot{\theta}\sin\theta)$$

$$|\dot{\mathbf{r}}|^2=\ell^2\dot{\theta}^2\cos^2\theta+\ell^2\dot{\theta}^2\sin^2\theta=\ell^2\dot{\theta}^2$$

$h = Y = l(1-\cos\theta)$ 　 hとv²に代入.

$\therefore L = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$ //

(2) $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}$

$\Rightarrow \frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$ 　 $\dot{\theta}$ を変数として扱って下さい

$(RHS) = \frac{d}{dt}(ml^2\dot{\theta})$

$= ml^2\ddot{\theta}$

$(LHS) = -mgl\sin\theta$

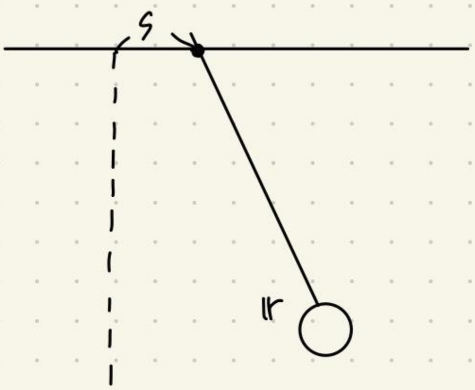
$ml^2\ddot{\theta} = -mgl\sin\theta$

$l\ddot{\theta} = -g\sin\theta$ // 　 $\simeq -g\theta$

$\theta = A\sin(\sqrt{\frac{g}{l}} + \alpha)$

$\omega = \sqrt{\frac{l}{g}}$

(3)



$r = (l\sin\theta + s, l(1-\cos\theta))$

sはtの関数

$v = \dot{r} = (-l\dot{\theta}\cos\theta + \dot{s}, l\dot{\theta}\sin\theta)$

$|v|^2 = l^2\dot{\theta}^2 + (\dot{s})^2 + 2l\dot{s}\dot{\theta}\cos\theta$

$L = \frac{1}{2}mv^2 - mgh$

$= \frac{1}{2}m\left(l^2\dot{\theta}^2 + (\dot{s})^2 + 2l\dot{s}\dot{\theta}\cos\theta\right) - mgl(1-\cos\theta)$

$\Rightarrow \frac{\partial L}{\partial \theta} = \frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}}$

$(RHS) = \frac{d}{dt}\left(ml^2\dot{\theta} + ml\underbrace{\dot{s}}_{\text{tの変数}}\cos\theta\right)$

$= ml^2\ddot{\theta} + ml\ddot{s}\cos\theta - ml\dot{\theta}\dot{s}\sin\theta$

$(LHS) = -ml\dot{s}\dot{\theta}\sin\theta - mgl\sin\theta$

$ml^2\ddot{\theta} + ml\ddot{s}\cos\theta = -mgl\sin\theta$

$l\ddot{\theta} = -\ddot{s}\cos\theta - g\sin\theta \quad \cdots (*)$ //

(4) $X = a\sin\omega t$, $Y = 0$: 支点.

$s = a\sin\omega t$. $(*)$に代入

$\sin\theta \approx \theta$, $\cos\theta \approx 1$

$l\ddot{\theta} + \ddot{s} + g\theta = 0$

$l\ddot{\theta} + g\theta = a\omega^2\sin\omega t$ … 非同次

$l\ddot{\theta} + g\theta = 0$ … 同次

$\theta = A\sin\left(\sqrt{\frac{g}{l}}t + \alpha\right)$

(A, αは定数)

$\theta = B\sin\omega t$ と仮定

非同次項がsinだから 2階微分で消えず

$-l\omega^2 B + gB = a\omega^2$

$B = \frac{a\omega^2}{g - l\omega^2}$

$\theta = \frac{a\omega^2}{g - l\omega^2}\sin\omega t$ 　 は特解

一般解

$$\theta = A\sin\left(\sqrt{\frac{g}{l}}t+\alpha\right)+\frac{a\omega^2}{g-l\omega^2}\sin\omega t$$

(5)

$$\theta = A\sin\left(\underbrace{\sqrt{\frac{g}{l}}}_{=\omega_0}t+\alpha\right)+\frac{a\omega^2}{g-l\omega^2}\sin\omega t$$

$\omega_0 > \omega$ と $\omega_0 < \omega$