2016 東工大 材料 数学

(1) $x = r\tan\theta$ $\qquad x : -a \to a$

$\tan\theta = \frac{x}{r}$ $\qquad \theta : \mathrm{ArcTan}\frac{x}{a} \to \mathrm{ArcTan}\frac{x}{a}$

$\theta = \mathrm{ArcTan}\frac{x}{r}$

$dx = \frac{d\theta}{\cos^2\theta}$

$$\int_{-\mathrm{Arc}}^{\mathrm{Arc}} \frac{r}{(r^2+r^2\tan^2\theta)^{\frac{3}{2}}} \frac{d\theta}{\cos^2\theta}$$

$$= \int_{-\mathrm{Arc}}^{\mathrm{Arc}} \frac{r}{r^3} \frac{1}{(1+\tan^2\theta)^{\frac{3}{2}}} \frac{1}{\cos^2\theta}\, d\theta$$

$$= \int_{\mathrm{Arc}}^{\mathrm{Arc}} \frac{1}{r^2} \cos^3\theta \frac{1}{\cos^2\theta}\, d\theta$$

$$= \int_{-\mathrm{Arc}}^{\mathrm{Arc}} \frac{1}{r^2}\cos\theta\, d\theta = \frac{1}{r^2}\Big[\sin\theta\Big]_{-\mathrm{Arc}}^{\mathrm{Arc}}$$

$$= \frac{2}{r^2}\sin\left(\mathrm{ArcTan}\frac{a}{r}\right)$$

(2) $\frac{dx}{dt} = k\{ab-(a+b)x+x^2\}$

$\frac{dx}{dt} = kab - k(a+b)x + kx^2$

$\frac{dx}{dt} = k(a-x)(b-x)$

$\frac{dx}{(a-x)(b-x)} = k\,dt$

$$\frac{1}{(a-x)(b-x)} = \frac{A}{(a-x)} + \frac{B}{(b-x)}$$

$$1 = Ab - Ax + aB - Bx$$
$$= -(A+B)x + bA + aB$$

$\begin{cases} A+B=0 \\ bA+aB=1 \end{cases}$ $\qquad (b-a)A = 1$

$A = \frac{1}{b-a},\ B = \frac{1}{a-b}$

$$\frac{1}{(a-x)(b-x)} = \frac{1}{(a-x)(b-a)} + \frac{1}{(b-x)(a-b)}$$

$$\frac{1}{(a-b)}\int \frac{1}{b-x}dx + \frac{1}{b-a}\int \frac{1}{a-x}\, dx = kt + C$$

$$\frac{-1}{a-b}\log b-x + \frac{1}{a-b}\log a-x = kt + C$$

$$\frac{1}{(a-b)}\log\frac{a-x}{b-x} = kt + C$$

$$\frac{a-x}{b-x} = De^{(a-b)kt}$$

$$a-x = Dbe^{(a-b)kt} - Dxe^{(a-b)kt}$$

$$(De^{(a-b)kt} - 1)x = Dbe^{(a-b)kt} - a$$

$$x = \frac{Dbe^{(a-b)kt} - a}{De^{(a-b)kt} - 1}$$

$$x(0) = \frac{Db-a}{D-1} = 0$$

$$Db = a$$

$$D = \frac{a}{b}$$

$$x = \frac{a e^{(a-b)kt} - a}{\frac{a}{b} e^{(a-b)kt} - 1} = \underline{\frac{ab e^{(a-b)kt} - ab}{a e^{(a-b)kt} - b}}$$

(3) $\det|A - \lambda I| = \det\begin{vmatrix} 1-\lambda & 0 & -2 \\ 1 & 2-\lambda & 2 \\ 1 & 1 & -\lambda \end{vmatrix}$

$$= -(1-\lambda)(2-\lambda)\lambda - 2 + 2(2-\lambda) - 2(1-\lambda)$$

$$= -(1-\lambda)(2-\lambda)\lambda + 4 - 2 - 2 - 2\lambda + 2\lambda$$

$$= -(1-\lambda)(2-\lambda)\lambda = 0$$

$$\lambda = 1, 2, 0$$

$\lambda = 0$ のとき $\begin{pmatrix} 1 & 0 & -2 \\ 1 & 2 & 2 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x-2z \\ x+2y+2z \\ x+y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x = k$ とすると.

$z = \frac{1}{2}k$

$y = -k$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix}$$

$$\sqrt{1^2 + (-1)^2 + \frac{1}{4}} = \sqrt{\frac{9}{4}} = \frac{3}{2}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{2}{3}\begin{pmatrix} 1 \\ -1 \\ \frac{1}{2} \end{pmatrix} = \underline{\frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}$$

$\lambda = 1$ のとき

$$\begin{pmatrix} 0 & 0 & -2 \\ 1 & 1 & 2 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2z \\ x+y+2z \\ x+y-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$z = 0$, $x = k$ とすると, $y = -k$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \qquad k = \frac{1}{\sqrt{1^2 + (-1)^2}} = \frac{1}{\sqrt{2}}$$

$$= \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}}$$

$\lambda = 2$ のとき

$$\begin{pmatrix} -1 & 0 & -2 \\ 1 & 0 & 2 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -x-2z \\ x+2z \\ x+y-2z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$z = k$ とすると

$x = -2k$

$y = 2k + 2k = 4k$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

$$k = \frac{1}{\sqrt{(-2)^2 + 4^2 + 1^2}} = \underline{\frac{1}{3\sqrt{3}}}$$

$$u_1 = \frac{1}{3}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}, \quad u_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \quad u_3 = \frac{1}{3\sqrt{3}}\begin{pmatrix} -2 \\ 4 \\ 1 \end{pmatrix}$$

$$3\sqrt{3} \times \sqrt{2} = 3\sqrt{6}$$

$$u_1 = \frac{1}{3\sqrt{6}}\begin{pmatrix} 2\sqrt{6} \\ -2\sqrt{6} \\ \sqrt{6} \end{pmatrix}, \quad u_2 = \frac{1}{3\sqrt{6}}\begin{pmatrix} 3\sqrt{3} \\ -3\sqrt{3} \\ 0 \end{pmatrix}, \quad u_3 = \frac{1}{3\sqrt{6}}\begin{pmatrix} -2\sqrt{2} \\ 4\sqrt{2} \\ \sqrt{2} \end{pmatrix}$$

$$P = \frac{1}{3\sqrt{6}}\begin{pmatrix} 2\sqrt{6} & 3\sqrt{3} & -2\sqrt{2} \\ -2\sqrt{6} & -3\sqrt{3} & 4\sqrt{2} \\ \sqrt{6} & 0 & \sqrt{2} \end{pmatrix}$$

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$$\left(\begin{array}{ccc|ccc} 2\sqrt{6} & 3\sqrt{3} & -2\sqrt{2} & 1 & 0 & 0 \\ -2\sqrt{6} & -3\sqrt{3} & 4\sqrt{2} & 0 & 1 & 0 \\ \sqrt{6} & 0 & \sqrt{2} & 0 & 0 & 1 \end{array}\right) = \left(\begin{array}{ccc|ccc} 2\sqrt{6} & 3\sqrt{3} & -2\sqrt{2} & 1 & 0 & 0 \\ 0 & 0 & 2\sqrt{2} & 1 & 1 & 0 \\ \sqrt{6} & 0 & \sqrt{2} & 0 & 0 & 1 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} \sqrt{6} & 0 & \sqrt{2} & 0 & 0 & 1 \\ 0 & 0 & 2\sqrt{2} & 1 & 1 & 0 \\ 0 & 3\sqrt{3} & -4\sqrt{2} & 1 & 0 & -2 \end{array}\right) = \left(\begin{array}{ccc|ccc} \sqrt{6} & 0 & \sqrt{2} & 0 & 0 & 1 \\ 0 & 3\sqrt{3} & 0 & 3 & 2 & -2 \\ 0 & 0 & 2\sqrt{2} & 1 & 1 & 0 \end{array}\right)$$

$$= \left(\begin{array}{ccc|ccc} \sqrt{6} & 0 & 0 & -\frac{1}{2} & -\frac{1}{2} & 1 \\ 0 & 3\sqrt{3} & 0 & 3 & 2 & -2 \\ 0 & 0 & 2\sqrt{2} & 1 & 1 & 0 \end{array}\right) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{1}{2\sqrt{6}} & -\frac{1}{2\sqrt{6}} & \frac{1}{\sqrt{6}} \\ 0 & 1 & 0 & \frac{1}{\sqrt{3}} & \frac{2}{3\sqrt{3}} & -\frac{2}{3\sqrt{3}} \\ 0 & 0 & 1 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 \end{array}\right)$$

$$P^{-1} = \frac{1}{\sqrt{6}}\begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} & 1 \\ \sqrt{2} & \frac{2}{3}\sqrt{2} & -\frac{2}{3}\sqrt{2} \\ \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \end{pmatrix} = \frac{1}{6\sqrt{6}}\begin{pmatrix} -3 & -3 & 6 \\ 6\sqrt{2} & 4\sqrt{2} & -4\sqrt{2} \\ 3\sqrt{3} & 3\sqrt{3} & 0 \end{pmatrix}$$

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(6) $P^{-1}AP = B$ $\qquad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$P^{-1}A^nP = B^n$ $\qquad B^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}$

$PP^{-1}A^nPP^{-1} = PB^nP^{-1}$

$A^n = PB^nP^{-1}$

$$PB^n = \frac{1}{3\sqrt{6}}\begin{pmatrix} 2\sqrt{6} & 3\sqrt{3} & -2\sqrt{2} \\ -2\sqrt{6} & -3\sqrt{3} & 4\sqrt{2} \\ \sqrt{6} & 0 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^n \end{pmatrix}$$

$$= \frac{1}{3\sqrt{6}}\begin{pmatrix} 0 & 3\sqrt{3} & -2^{n+1}\sqrt{2} \\ 0 & -3\sqrt{3} & 2^{n+2}\sqrt{2} \\ 0 & 0 & 2^n\sqrt{2} \end{pmatrix}$$

$$PB^nP^{-1} = \frac{1}{3\sqrt{6}} \cdot \frac{1}{6\sqrt{6}}\begin{pmatrix} 0 & 3\sqrt{3} & -2^{n+1}\sqrt{2} \\ 0 & -3\sqrt{3} & 2^{n+2}\sqrt{2} \\ 0 & 0 & 2^n\sqrt{2} \end{pmatrix}\begin{pmatrix} -3 & -3 & 6 \\ 6\sqrt{2} & 4\sqrt{2} & -4\sqrt{2} \\ 3\sqrt{3} & 3\sqrt{3} & 0 \end{pmatrix}$$

$$= \frac{1}{108}\begin{pmatrix} 18\sqrt{6} - 2^{n+1} \cdot 3\sqrt{6} & 12\sqrt{6} - 3 \cdot 2^{n+1}\sqrt{6} & \overset{-12\sqrt{6}}{\cancel{2^{n+3}\,2}} \\ -18\sqrt{6} + 2^{n+2} \cdot 3\sqrt{6} & -12\sqrt{6} + 3 \cdot 2^{n+2}\sqrt{6} & 12\sqrt{6} \\ 3 \cdot 2^n\sqrt{6} & 3 \cdot 2^n\sqrt{6} & 0 \end{pmatrix}$$

$$= \frac{\sqrt{6}}{108}\begin{pmatrix} 18 - 2^{n+1} \cdot 3 & 12 - 3 \cdot 2^{n+1} & \cancel{2^{n+4}\frac{1}{\sqrt{6}}} -12 \\ -18 + 3 \cdot 2^{n+2} & -12 + 3 \cdot 2^{n+2} & 12 \\ 3 \cdot 2^n & 3 \cdot 2^n & 0 \end{pmatrix}$$

$$= \frac{3\sqrt{6}}{108}\begin{pmatrix} 6 - 2^{n+1} & 4 - 2^{n+1} & -4 \\ -6 + 2^{n+2} & -4 + 2^{n+2} & 4 \\ 2^n & 2^n & 0 \end{pmatrix}$$

$$= \frac{6\sqrt{6}}{\cancel{108}\,18}\begin{pmatrix} 3 - 2^n & 2 - 2^n & -2 \\ -3 + 2^{n+1} & -2 + 2^{n+1} & 2 \\ 2^{n-1} & 2^{n-1} & 0 \end{pmatrix}$$

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(2) (3) $\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 6y = \sin x$

$y'' + 5y' + 6y = 0 \qquad (\lambda + 3)(\lambda + 2) = 0$

$\lambda = -2, -3$

$y = Ae^{-2x} + Be^{-3x}$

$y = C\sin x + D\cos x$ とおく.

$$-C\sin x - D\cos x + 5C\cos x - 5D\sin x + 6C\sin x + 6D\cos x = \sin x$$

$$(-C-5D+6C)\sin x + (-D+5C+6D)\cos x = \sin x$$

$$5(C-D)\sin x + (D+C)5\cos x = \sin x$$

$$\begin{cases} C-D = \frac{1}{5} \\ D+C = 0 \end{cases} \quad 2C = \frac{1}{5},\ C = \frac{1}{10}$$

$$D = -\frac{1}{10}$$

よって一般解は

$$y = Ae^{-2x} + Ce^{-3x} + \frac{1}{10}\sin x - \frac{1}{10}\cos x$$

(7) $f(x)^2 = \frac{a_0^2}{4} + \sum_n a_0(a_n\cos nx + b_n\sin nx)$

$$+\sum_n\sum_m \Big(a_n a_m\cos nx\cos mx + a_n b_m\cos nx\sin mx + a_m b_n\cos mx\sin nx + b_n b_m\sin nx\sin mx\Big)$$

$n=m$ のときのみ残るため.

$$f(x)^2 = \frac{a_0^2}{4} + \sum_n a_0(a_n\cos nx + b_n\sin nx) + \sum_n\left(a_n^2\cos^2 nx + 2a_n b_n\cos nx\sin nx + b_n^2\sin^2 nx\right)$$

両辺 $-\pi \to \pi$ で積分すると.

$$\int_{-\pi}^{\pi}|f(x)|^2 dx = \int\frac{a_0^2}{4}dx + \sum a_0\int a_n\cos nx + b_n\sin nx\, dx + \sum\left(\int a_n^2\cos^2 nx + 2a_n b_n\cos nx\sin nx + b_n^2\sin^2 nx\, dx\right)$$

$$\int_{-\pi}^{\pi}\cos nx\, dx = \int_{-\pi}^{\pi}\sin nx\, dx = \int_{-\pi}^{\pi}\cos nx\sin nx = 0$$

$$\int_{-\pi}^{\pi}|f(x)|^2 dx = \frac{a_0^2}{2}\pi + \sum a_0 \times 0 + \sum\left(\int a_n^2\frac{1+\cos 2nx}{2} + b_n^2\frac{1-\cos 2nx}{2}\, dx\right)$$

$$= \frac{a_0^2\pi}{2} + \sum\left(a_n^2\pi + b_n^2\pi\right)$$

$$\therefore \frac{1}{\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx = \frac{a_0^2}{2} + \sum_n\left(a_n^2 + b_n^2\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}x^2 dx = \frac{2}{3}\pi^2$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}x^2\cos nx\, dx = \frac{2}{\pi}\left\{\left[\frac{1}{n}\sin nx\, x^2\right]_0^{\pi} - \int\frac{2}{n}x\sin nx\, dx\right\}$$

$$= \frac{2}{\pi}\left(-\frac{2}{n}\right)\left\{\left[-\frac{x}{n}\cos nx\right]_0^{\pi} - \int_0^{\pi}-\frac{1}{n}\cos nx\, dx\right\}$$

$$= -\frac{4}{n\pi}\left(-\frac{1}{n}\right)\pi\cos n\pi \quad (n = 0, 1, 2\cdots)$$

$$= \frac{4\pi}{n^2\pi}(-1)^n$$

$$b_n = 0$$

$$\int_{-\pi}^{\pi}x^4 dx = \frac{1}{5}(\pi^5 + \pi^5) = \frac{2}{5}\pi^5$$

$$\therefore \frac{2}{5}\pi^4 = \frac{1}{2}\left(\frac{2}{3}\pi^2\right)^2 + \sum\frac{16}{n^4}(-1)^{2n}$$

$$\sum\frac{1}{n^4} = \frac{\pi^4}{40} - \frac{\pi^4}{72} = \frac{\pi^4}{90}$$