2016 東工大 材料 量子

(1) $\hat{p}\varphi = \left(-i\hbar\frac{\partial}{\partial x}\right)\exp(i(kx-\omega t)) = -i\hbar(ik)\varphi$

$= \hbar k\varphi$

$\hbar k$ は運動量 より $-i\hbar\frac{\partial}{\partial x}$ は運動量演算子。

(2) $\left[x, -i\hbar\frac{\partial}{\partial x}\right]\varphi$

$= -i\hbar x\frac{\partial\varphi}{\partial x} + i\hbar\frac{\partial}{\partial x}(x\varphi) = \cancel{-i\hbar x\frac{\partial\varphi}{\partial x}} + i\hbar\varphi + \cancel{i\hbar x\frac{\partial\varphi}{\partial x}} = i\hbar\varphi$

よって $\left[x, -i\hbar\frac{\partial}{\partial x}\right] = i\hbar$

(3) $\frac{d}{dt}\langle\psi(t)|\hat{A}|\psi(t)\rangle = \frac{d}{dt}\langle\psi(0)|e^{\frac{i\hat{H}t}{\hbar}}\hat{A}e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$

$= \langle\psi(0)|\frac{i\hat{H}}{\hbar}e^{\frac{i\hat{H}t}{\hbar}}\hat{A}e^{-\frac{i\hat{H}t}{\hbar}} + e^{\frac{i\hat{H}t}{\hbar}}\hat{A}\left(-\frac{i\hat{H}}{\hbar}\right)e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$

$= \frac{i}{\hbar}\langle\psi(0)|e^{\frac{i\hat{H}t}{\hbar}}\hat{H}\hat{A}e^{-\frac{i\hat{H}t}{\hbar}} - e^{\frac{i\hat{H}t}{\hbar}}\hat{A}\hat{H}e^{-\frac{i\hat{H}t}{\hbar}}|\psi(0)\rangle$

$= \frac{-i}{\hbar}\langle\psi(t)|[\hat{A},\hat{H}]|\psi(t)\rangle$

$= \frac{1}{i\hbar}\langle[\hat{A},\hat{H}]\rangle$

(4) $\frac{d}{dt}\langle x\rangle = \frac{d}{dt}\int\psi^*x\psi\,dx$

$= \int\frac{d\psi^*}{dt}x\psi + \psi^*x\frac{d\psi}{dt}\,dx$

$$\begin{cases} i\hbar\frac{d}{dt}\psi = -\frac{\hbar^2}{2m}\psi'' + V\psi \to \frac{d}{dt}\psi = -\frac{\hbar}{2mi}\psi'' + \frac{1}{i\hbar}V\psi \\ -i\hbar\frac{d}{dt}\psi^* = -\frac{\hbar^2}{2m}\psi''^* + V\psi^* \to \frac{d}{dt}\psi^* = \frac{\hbar}{2mi}\psi''^* - \frac{1}{i\hbar}V\psi^* \end{cases}$$

$\frac{1}{i\hbar}\times 2mi = \frac{2m}{\hbar} = A$ とおく。

より

$= \frac{\hbar}{2mi}\int\psi''^*x\psi - \psi^*x\psi''\,dx + \cancel{AV\psi^*x\psi - AV\psi^*x\psi}\,dx$

$= \frac{\hbar}{2mi}\int\psi'^*\psi - \cancel{\psi'^*\psi} + \psi^*\psi' + \cancel{\psi'^*\psi}\,dx$

$= \frac{\hbar}{2mi}\int\psi^*\psi' + \psi^*\psi'\,dx = \frac{\hbar}{mi}\int\psi^*\frac{\partial}{\partial x}\psi\,dx$

$= \frac{1}{m}\int\psi^*\left(-i\hbar\frac{\partial}{\partial x}\right)\psi\,dx$

$= \frac{1}{m}\langle p\rangle$

$\therefore\ m\frac{d}{dt}\langle x\rangle = \langle p\rangle$

(5) $\hbar\omega(a^\dagger a + \frac{1}{2})$ を考える。

(6) $\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\hat{x} + \sqrt{\frac{1}{2\hbar m\omega}}i\hat{p} = A\hat{x} + Bi\hat{p}$ とおく。

$\hbar\omega(a^\dagger a + \frac{1}{2}) = \hbar\omega\{(A\hat{x} - iB\hat{p})(A\hat{x} + iB\hat{p}) + \frac{1}{2}\}$

$= \hbar\omega\{A^2\hat{x}^2 + iAB\hat{x}\hat{p} - iAB\hat{p}\hat{x} + B^2\hat{p}^2 + \frac{1}{2}\}$

$= \hbar\omega(A^2\hat{x}^2 + iAB\cdot i\hbar + B^2\hat{p}^2 + \frac{1}{2})$

$= \hbar\omega\left(\frac{m\omega}{2\hbar}\hat{x}^2 + i\frac{1}{2\hbar}i\hbar + \frac{1}{2\hbar m\omega}\hat{p}^2 + \frac{1}{2}\right)$

$= \frac{m\omega^2}{2}x^2 + \frac{p^2}{2m} = \hat{H}$ と書ける。

$\hat{H} = \hbar\omega(\hat{n} + \frac{1}{2})|\psi\rangle = \hbar\omega(n + \frac{1}{2})|\psi\rangle$ より

固有値は $\hbar\omega(n + \frac{1}{2})$ $(n = 0, 1, 2 \dots)$

(7) $\hat{a}\psi_0 = 0$ より

$\left\{x + \frac{i}{m\omega}\left(-i\hbar\frac{\partial}{\partial x}\right)\right\}\psi_0 = 0$

$x\psi_0 + \frac{\hbar}{m\omega}\psi_0' = 0$

$\frac{\hbar}{m\omega}\frac{d\psi_0}{dx} = -x\psi_0$

$\frac{1}{\psi_0}\frac{\hbar}{m\omega}d\psi_0 = -x\,dx$

$\log\psi_0 = -\frac{m\omega}{2\hbar}x^2 + C$

$\psi_0 = D e^{-\frac{m\omega}{2\hbar}x^2}$

$\int\psi_0^*\psi_0\,dx = |D|^2\int e^{-\frac{m\omega}{\hbar}x^2}dx$

$= |D|^2\sqrt{\frac{\hbar\pi}{m\omega}} = 1$

$D = \left(\frac{m\omega}{\hbar\pi}\right)^{\frac{1}{4}}$

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$$\langle x\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} x\, e^{-\frac{m\omega}{\hbar}x^2}\,dx \quad = 0$$

$$\langle x^2\rangle = \sqrt{\frac{m\omega}{\pi\hbar}}\int_{-\infty}^{\infty} x^2 e^{-\frac{m\omega}{\hbar}x^2}\,dx$$

$$\left\{\begin{aligned} &\int e^{-ax^2}\,dx = \pi^{\frac{1}{2}}a^{-\frac{1}{2}} \\ &\int -\cancel{x}x^2 e^{-ax^2}\,dx = -\frac{1}{2}\pi^{\frac{1}{2}}a^{-\frac{3}{2}} \\ &\qquad \int x^2 e^{-ax^2}\,dx = \frac{\pi^{\frac{1}{2}}}{2}a^{-\frac{3}{2}} \end{aligned}\right\}$$

T'/

$$= \sqrt{\frac{m\omega}{\pi\hbar}}\,\frac{\sqrt{\pi}}{2}\left(\frac{\hbar}{m\omega}\right)^{\frac{3}{2}}$$

$$= \sqrt{\frac{m\omega}{\hbar}}\sqrt{\frac{\hbar}{m\omega}}\,\frac{\hbar}{m\omega} = \underline{\frac{\hbar}{m\omega}}$$