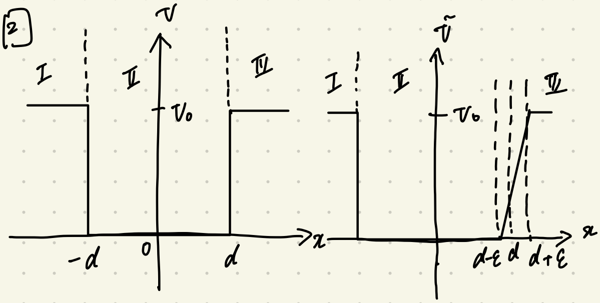
東工大院試 平成30年 午後

[2]



I : $\psi(x) = A_1 e^{\alpha x} + A_2 e^{-\alpha x}$

II : $\psi(x) = B_1 \sin\beta x + B_2 \cos\beta x$

III : $\psi(x) = C_1 e^{\alpha x} + C_2 e^{-\alpha x}$

(1) 波動関数の有界性

→ $x \to \pm\infty$ で $\psi \to 0$ になる.

I において

$x \to \pm\infty$,

$\psi(x) = A_1 e^{\alpha x} + \underline{A_2 e^{-\alpha x}}$ $(-\infty \le x \le -d)$

発散

$\psi(x) = \underline{C_1 e^{\alpha x}} + C_2 e^{-\alpha x}$ $(d \le x \le \infty)$

発散

このとき

$\psi(x) \to 0$ より

$A_2 = C_1 = 0$ //

$\psi_I''(x) = \alpha^2 \psi_I(x)$

$\psi_{II}''(x) = -\beta^2 B_1 \sin\beta x - \beta^2 B_2 \cos\beta x$

$= -\beta^2 \psi_{II}(x)$

$-\frac{\hbar^2}{2m}\psi''(x) + (V(x) - E)\psi(x) = 0$

$\frac{\hbar^2}{2m}\alpha^2 \psi_I(x) = (V - E)\psi_I(x)$

$\alpha^2 = \frac{2m(V-E)}{\hbar^2}$

$\alpha = \frac{\sqrt{2m(V-E)}}{\hbar^2}$

$\psi(II) = 0$ $(-d \le x \le d)$ より

$V_0 = 0$

$-\beta^2 \frac{\hbar^2}{2m}\psi_{II}(x) = -E\psi_{II}(x)$

$\beta^2 = \frac{2mE}{\hbar^2}$

$\beta = \frac{\sqrt{2mE}}{\hbar}$ //

両辺

↓

$\psi''(x) = \frac{2m}{\hbar^2}(V(x) - E)\psi(x)$

(両辺)を積分

$\int_{d-\varepsilon}^{d+\varepsilon} \psi''(x)\,dx = \left[\psi'(x)\right]_{d-\varepsilon}^{d+\varepsilon}$

$= \frac{d}{dx}\psi\Big|_{x=d+\varepsilon} - \frac{d}{dx}\psi\Big|_{x=d-\varepsilon}$ //

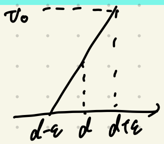
(右辺を積分)

$$\int_{d-\varepsilon}^{d+\varepsilon}\frac{2m}{\hbar^2}\left(\tilde{v}(x)-E\right)\varphi(x)\,dx$$

$$\simeq \frac{2m}{\hbar^2}\varphi(d)\left[\int_{d-\varepsilon}^{d+\varepsilon}\tilde{v}(x)\,dx-\int_{d-\varepsilon}^{d+\varepsilon}E\,dx\right]$$

今回は直線　　Eはx依存なしより

$$=\frac{2m}{\hbar^2}\varphi(d)\left[\int_{d-\varepsilon}^{d+\varepsilon}\frac{v_0}{2\varepsilon}x+\frac{v_0}{2\varepsilon}(\varepsilon-d)\ -2\varepsilon E\right] \quad \cdots(イ)$$



$\tilde{v}=\frac{v_0}{2\varepsilon}x+b$

$0=\frac{v_0}{2\varepsilon}(d-\varepsilon)+b$

$b=\frac{v_0}{2\varepsilon}(\varepsilon-d)$

$\therefore\ \tilde{v}=\frac{v_0}{2\varepsilon}x+\frac{v_0}{2\varepsilon}(\varepsilon-d)$

$$=\frac{2m}{\hbar^2}\varphi(d)\left\{\frac{v_0}{4\varepsilon}\left[(d+\varepsilon)^2-(d-\varepsilon)^2\right]+\frac{v_0}{2\varepsilon}(\varepsilon-d)\cdot2\varepsilon-2\varepsilon E\right\}$$

$$=\frac{2m}{\hbar^2}\varphi(d)\left\{\frac{v_0}{4\varepsilon}(4\varepsilon d)+v_0(\varepsilon-d)-2\varepsilon E\right\}$$

$$=\frac{2m}{\hbar^2}\varphi(d)\left(v_0 d+v_0(\varepsilon-d)-2\varepsilon E\right)$$

$$=\frac{2m}{\hbar^2}\varphi(d)\left(v_0\varepsilon-2\varepsilon E\right)$$

$$=\frac{2m}{\hbar^2}\varphi(d)\left(v_0-2E\right)\varepsilon$$

$\varepsilon\to0$

$\to 0 \quad \cdots(ウ)$

(3) $\varphi(x)$ が $x=-d,\ x=d$ において満たすべき条件.

$$\begin{cases}\varphi_{I}(-d)=\varphi_{II}(-d)\\ \varphi_{I}'(-d)=\varphi_{II}'(-d)\end{cases}\quad\cdots①$$

$$\begin{cases}\varphi_{II}(d)=\varphi_{III}(d)\\ \varphi_{II}'(d)=\varphi_{III}'(d\end{cases}\quad\cdots②$$

①について

$$A_1e^{-\alpha d}=-B_1\sin\beta d+B_2\cos\beta d$$

$$\alpha A_1e^{-\alpha d}=\beta\left(B_1\cos\beta d+B_2\sin\beta d\right)$$

②について

$$B_1\sin\beta d+B_2\cos\beta d=C_2e^{-\alpha d}$$

$$\beta\left(B_1\cos\beta d-B_2\sin\beta d\right)=-\alpha C_2e^{-\alpha d}$$

(4) $\tilde{\alpha}=\alpha d$

$\tilde{\beta}=\beta d$

$B_1\neq0$, $B_2\neq0$ の解が存在

②について

$$B_1\sin\beta d+B_2\cos\beta d=C_2e^{-\alpha d}$$

$$-)\ B_1\cdot-\frac{\beta}{\alpha}\cos\beta d+B_2\frac{\beta}{\alpha}\sin\beta d=C_2e^{-\alpha d}$$

$$B_1\left(\sin\tilde{\beta}+\frac{\beta}{\alpha}\cos\tilde{\beta}\right)+B_2\left(\cos\tilde{\beta}-\frac{\beta}{\alpha}\sin\tilde{\beta}\right)=0$$

$\cdots②'$

①について

$$A_1 e^{-\alpha d} = -B_1 \sin\beta d + B_2 \cos\beta d$$

$$\alpha A_1 e^{-\alpha d} = \beta(B_1 \cos\beta d + B_2 \sin\beta d)$$

↓

$$-)\ A_1 e^{-\alpha d} = \frac{\beta}{\alpha}(B_1 \cos\beta d + B_2 \sin\beta d)$$

$$0 = B_1\left(-\sin\tilde{\beta} - \frac{\beta}{\alpha}\cos\tilde{\beta}\right) + B_2\left(\cos\tilde{\beta} - \frac{\beta}{\alpha}\sin\tilde{\beta}\right)$$

$$B_1\left(\sin\tilde{\beta} + \frac{\beta}{\alpha}\cos\tilde{\beta}\right) + B_2\left(\frac{\beta}{\alpha}\sin\tilde{\beta} - \cos\tilde{\beta}\right) = 0 \quad \cdots ①'$$

①'と②'を行列に

$$\begin{pmatrix} \left(\frac{\tilde{\beta}}{\tilde{\alpha}}\cos\tilde{\beta} + \sin\tilde{\beta}\right) & \left(\frac{\tilde{\beta}}{\tilde{\alpha}}\sin\tilde{\beta} - \cos\tilde{\beta}\right) \\ \left(\frac{\tilde{\beta}}{\tilde{\alpha}}\cos\tilde{\beta} + \sin\tilde{\beta}\right) & \left(-\frac{\tilde{\beta}}{\tilde{\alpha}}\sin\tilde{\beta} + \cos\tilde{\beta}\right) \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = 0$$

$$= 2\left(\frac{\tilde{\beta}}{\tilde{\alpha}}\cos\tilde{\beta} + \sin\tilde{\beta}\right)\left(-\frac{\tilde{\beta}}{\tilde{\alpha}}\sin\tilde{\beta} + \cos\tilde{\beta}\right) = 0$$

$\tilde{\alpha} = -\tilde{\beta}\cot\tilde{\beta}$ 　　$\tilde{\alpha} = \tilde{\beta}\tan\tilde{\beta}$

(5) V_0が与えられたときの束縛状態の個数を

$\tilde{\gamma} = d\frac{\sqrt{2mV_0}}{\hbar}$ を用いて表す

→ $\tilde{\gamma}^2$を$\tilde{\alpha}$と$\tilde{\beta}$を用いて表す

$$\tilde{\alpha} = \frac{\sqrt{2m(V_0 - E)}}{\hbar} d$$

$$\tilde{\beta} = \frac{\sqrt{2mE}}{\hbar} d$$

$$\tilde{\alpha}^2 + \tilde{\beta}^2 = d^2\left(\frac{2m(V_0 - E)}{\hbar^2} + \frac{2mE}{\hbar^2}\right)$$

$$= d\left(\frac{2mV_0}{\hbar^2}\right)$$

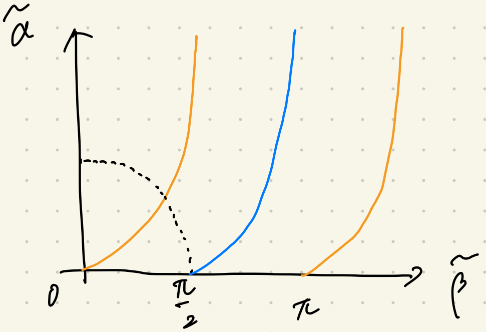
$$= \tilde{\gamma}^2$$

(4)で求めた式を使って$\tilde{\gamma}$についての式がある.

$$\tilde{\alpha} = -\tilde{\beta}\cot\tilde{\beta}$$

$$\tilde{\alpha} = \tilde{\beta}\tan\tilde{\beta}$$

$\tilde{\gamma}^2 = \tilde{\alpha}^2 + \tilde{\beta}^2$ と円の式



$\tilde{\gamma} = \frac{n\pi}{2}$ で状態が1つ増える

$\therefore\ n = \frac{2}{\pi}\tilde{\gamma}$

$\therefore\ \left[\frac{2\tilde{\gamma}}{\pi}\right]$ //

(6) $\tilde{\gamma} \ll 1$

基底状態のエネルギーをE_0

$$\frac{V_0 - E_0}{V_0} = \frac{1}{V_0}\cdot\frac{\hbar^2\alpha^2}{2m}$$

$$\alpha^2 = \frac{2m(V_0 - E)}{\hbar^2}$$

$$V_0 - E = \frac{\hbar^2\alpha^2}{2m}$$ より

$$= \frac{\hbar^2}{d^2\, 2mV_0}\alpha^2 d^2$$

$$\tilde{\gamma} = \frac{d\sqrt{2mV_0}}{\hbar}$$

$$\tilde{\gamma}^2 = \frac{d^2\, 2mV_0}{\hbar^2}$$ より

$$= \frac{\tilde{\alpha}^2}{\tilde{\gamma}^2}$$

$\tilde{\gamma} \ll 1 \to \tilde{\alpha}, \tilde{\beta} \ll 1$

$$\tilde{\alpha} = \tilde{\beta}\tan\tilde{\beta}$$

$$\approx \tilde{\beta}\tilde{\beta}$$

$$\tan\theta = \theta\cdot\frac{\theta!}{3}\cdots$$

$(\theta \ll 1)$

$$= \tilde{\beta}^2$$

$\tilde{\alpha} \doteqdot \tilde{\beta}^2$ より

$$\tilde{\alpha} \ll \tilde{\beta}$$

$$\tilde{\gamma}^2 = \tilde{\alpha}^2 + \tilde{\beta}^2 \approx \tilde{\beta}^2$$

$$\frac{\tilde{\alpha}^2}{\tilde{f}^2} \approx \frac{\tilde{\beta}^4}{\tilde{f}^2} \approx \frac{\tilde{f}^4}{\tilde{f}^2}$$

$$\approx \tilde{f}^2$$