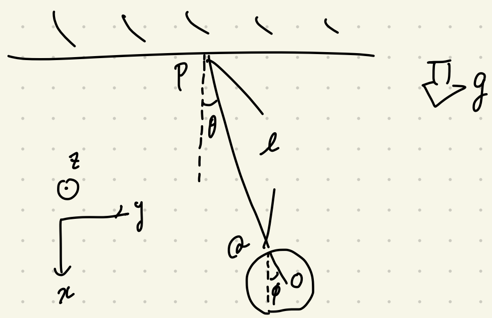
平成31年度　東工大院試　午前

[1]



M：円板の質量　m：棒の質量

(1) P→z方向に沿った軸まわり I_l

O→z　〃　I_a

I_l は P で回転

$$I_l = \int_0^l \rho x^2 \, dx = \frac{l^3\rho}{3}$$

$\rho l = m$ より

$$= \frac{ml^2}{3}$$

$$I_a = \int_0^a r^2 \, dM = \int_0^a \rho' 2\pi r^3 \, dr$$

$$M = r^2\pi\rho'$$

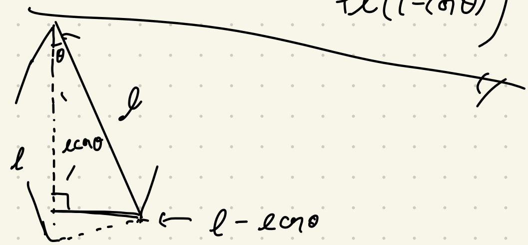
$dM = 2\pi r \cdot \rho' \, dr$ より

$$= \frac{\rho'\pi a^4}{2}$$

$M = \pi a^2 \rho'$ より $= \frac{M}{2}a^2$

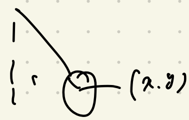
(2) 振り子全体のポテンシャルエネルギー

$$V = mg\frac{l}{2}(1-\cos\theta) + Mg\{a(1-\cos\phi) + l(1-\cos\theta)\}$$



← $l - l\cos\theta$
$= l(1-\cos\theta)$

(3) 全体の運動エネ



とすると

$$\begin{cases} x = l\cos\theta + a\cos\phi \\ y = l\sin\theta + a\sin\phi \end{cases}$$

$$\rightarrow \begin{cases} \dot x = -(l\dot\theta\sin\theta + a\dot\phi\sin\phi) \\ \dot y = l\dot\theta\cos\theta + a\dot\phi\cos\phi \end{cases}$$

$I = I_0 + T_G$
I：固定軸での回転エネ
I_0, T_G：重心での回転エネ 並進エネ

$$T = \underbrace{\frac12 I_l\dot\theta^2}_{P\text{まわりエネ}} + \underbrace{\frac12 I_a\dot\phi^2}_{Q\text{まわりエネ}}$$

$$= \frac12 I_l\dot\theta^2 + \frac12 M(\dot x^2+\dot y^2) + \frac12 I_a\dot\phi^2$$

固定軸まわりでの回転エネ ↔ 重心での並進…

$$\dot x^2 + \dot y^2 = (l\dot\theta)^2 + (a\dot\phi)^2 + 2al\dot\theta\dot\phi\underbrace{(\cos\theta\cos\phi + \sin\theta\sin\phi)}_{\cos(\theta-\phi)}$$

よって

$$T = \frac{ml^2}{6}\dot\theta^2 + \frac{Ma^2}{4}\dot\phi^2 + \frac{M}{2}\{(l\dot\theta)^2 + (a\dot\phi)^2 + 2al\dot\theta\dot\phi\cos(\theta-\phi)\}$$

$= 1 - \frac12(\theta-\phi)^2$

消える 微小量 4次

(4) ラグランジアン → オイラー・ラグランジュ

$L = T - V$　　$\dot\theta, \dot\phi$ と 微小量

ex) $\cos\theta \simeq 1 - \frac12\theta^2$

$$T \approx \underbrace{\frac{ml^2}{6}\dot{\theta}^2 + \frac{Ma^2}{4}\dot{\phi}^2}_{} + \frac{M}{2}\underbrace{\{\underbrace{(l\dot{\theta})^2 + (a\dot{\phi}^2)}_{} + 2al\dot{\theta}\dot{\phi}\}}_{(l\dot{\theta}+a\dot{\phi})^2}$$

$$V \approx \frac{mg}{4}l\theta^2 + \frac{Mg}{2}\{a\phi^2 + l\theta^2\}$$

$\therefore L = T - V$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta}$ と $\frac{d}{dt}\frac{\partial L}{\partial \dot{\phi}} - \frac{\partial L}{\partial \phi}$ を求める.

$$L = \frac{ml^2}{6}\dot{\theta}^2 + \frac{Ma^2}{4}\dot{\phi}^2 + \frac{M}{2}(l\dot{\theta} + a\dot{\phi})^2 - \frac{mg}{4}l\theta^2 + \frac{Mg}{2}\{a\phi^2 + l\theta^2\}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{d}{dt}\left(\frac{ml^2}{3}\dot{\theta} + M(l\dot{\theta} + a\dot{\phi}) \times l\right)$$

$$= \frac{ml^2}{3}\ddot{\theta} + Ml(l\ddot{\theta} + a\ddot{\phi})$$

$$= \left(\frac{m}{3} + M\right)l^2\ddot{\theta} + Mal\ddot{\phi}$$

$$-\frac{\partial L}{\partial \theta} = -\left(\frac{mg}{2}l\theta + Mgl\theta\right)$$

$$= l\theta\left(\frac{mg}{2} + Mg\right)$$

$$= lg\theta\left(\frac{m}{2} + M\right)$$

$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$ より

$$\underline{\left(\frac{m}{3} + M\right)l\ddot{\theta} + Ma\ddot{\phi} = -g\theta\left(\frac{m}{2} + M\right)}$$

ϕ も同様に計算すると

$$l\ddot{\theta} + \frac{3}{2}a\ddot{\phi} = -g\phi$$

(5) オイラー・ラグランジュの解である

2種類の角振動数 ω_+, ω_-

$\frac{m}{M} \to 0$ での ω_+^2, ω_-^2 を求める

円振動!

$$\begin{cases} \left(\frac{m}{3} + M\right)l\ddot{\theta} + Ma\ddot{\phi} = -g\theta\left(\frac{m}{2} + M\right) \\ l\ddot{\theta} + \frac{3}{2}a\ddot{\phi} = -g\phi \end{cases}$$

$$M\left(\frac{m}{3M} + 1\right)l\ddot{\theta} + Ma\ddot{\phi} = -g\theta M\left(\frac{m}{2M} + 1\right)$$

$$Ml\ddot{\theta} + Ma\ddot{\phi} = -Mg\theta$$

$$\therefore \begin{cases} l\ddot{\theta} + a\ddot{\phi} = -g\theta \\ l\ddot{\theta} + \frac{3}{2}a\ddot{\phi} = -g\phi \end{cases}$$

$\theta = A\sin\omega t$, $\phi = B\sin\omega t$ とおく.

$$\therefore -Al\omega^2\sin\omega t - Ba\omega^2\sin\omega t = -g\theta$$

$$-lA\omega^2\sin\omega t - B\frac{3}{2}a\omega^2\sin\omega t = -g\phi$$

$$\omega_0 = \sqrt{\frac{g}{l}}$$

$$\begin{cases} A\omega^2\sin\omega t + B\frac{a}{l}\omega^2\sin\omega t = \frac{g}{l}\theta \\ A\omega^2\sin\omega t + B\frac{3}{2}\frac{a}{l}\omega^2\sin\omega t = \frac{g}{l}\phi \end{cases}$$

$$\begin{cases} A\omega^2 \sin\omega t - \omega_0^2\theta + B\frac{a}{l}\omega^2 \sin\omega t = 0 \\ A\omega^2 \sin\omega t - \omega_0^2\phi + B\frac{a}{l}\frac{3}{2}\omega^2 \sin\omega t = 0 \end{cases}$$

$$\begin{cases} A\omega^2 \sin\omega t - \omega_0^2 A\sin\omega t + B\frac{a}{l}\omega^2 \sin\omega t = 0 \\ A\omega^2 \sin\omega t - \omega_0^2 B\sin\omega t + B\frac{3}{2}\frac{a}{l}\omega^2 \sin\omega t = 0 \end{cases}$$

$$\begin{cases} A(\omega^2 - \omega_0^2) + B\frac{a}{l}\omega^2 = 0 \\ A\omega^2 + B\frac{a3}{l2}\left(\omega^2 - \frac{2l}{3a}\omega_0^2\right) = 0 \end{cases}$$

$$\begin{pmatrix} \omega^2 - \omega_0^2 & \frac{a}{l}\omega^2 \\ \omega^2 & \frac{3}{2}\frac{a}{l}\omega^2 - \omega_0^2 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$A, B = 0$ は自明 → 非自明解を…

$$(\omega^2 - \omega_0^2)\left(\frac{3}{2}\frac{a}{l}\omega^2 - \omega_0^2\right) - \frac{a}{l}\omega^4$$

$$= \frac{3}{2}\frac{a}{l}\omega^4 - \omega^2\omega_0^2 - \frac{3}{2}\frac{a}{l}\omega^2\omega_0^2 + \omega_0^4 - \frac{a}{l}\omega^4$$

$$= \frac{1}{2}\frac{a}{l}\omega^4 - \left(\frac{3}{2}\frac{a}{l} + 1\right)\omega^2\omega_0^2 + \omega_0^4 = 0$$

$\omega^2 = x$ とおく.

F根 → ω_0^2

$$\frac{1}{2}\frac{a}{l}x^2 - \omega_0^2\left(\frac{3}{2}\frac{a}{l} + 1\right)x + \omega_0^4 = 0$$

$$x = \frac{1}{\frac{a}{l}}\left\{\omega_0^2\left(\frac{3}{2}\frac{a}{l} + 1\right) \pm \sqrt{\omega_0^4\left(\frac{3}{2}\frac{a}{l} + 1\right)^2 - 4\frac{a}{2l}\omega_0^4}\right\}$$

$$= \frac{l}{a}\left\{\omega_0^2\left(\frac{3}{2}\frac{a}{l} + 1\right) \pm \sqrt{\omega_0^4\left(\frac{9a^2}{4l^2} + \frac{3a}{l} + 1\right) - \frac{2a}{l}\omega_0^4}\right\}$$

$$= \frac{l}{a}\left\{\omega_0^2\left(\frac{3}{2}\frac{a}{l} + 1\right) \pm \sqrt{\left(\frac{9a^2}{4l^2} + \frac{a}{2l} + 1\right)\omega_0^4}\right\}$$

$$= \frac{l}{a}\left\{\omega_0^2\left(\frac{3}{2}\frac{a}{l} + 1\right) \pm \omega_0^2\sqrt{\frac{9a^2}{4l^2} + \frac{a}{2l} + 1}\right\}$$

$$= \frac{l}{a}\left\{\frac{g}{l}\left(\frac{3}{2}\frac{a}{l} + 1\right) \pm \frac{g}{l}\sqrt{\frac{9a^2}{4l^2} + \frac{a}{2l} + 1}\right\}$$

$$= \frac{g}{a}\left(\frac{3}{2}\frac{a}{l} + 1\right) \pm g\sqrt{\frac{9}{4l^2} + \frac{a}{2l} + 1}$$

$$= g\left(\frac{3}{2l} + \frac{1}{a}\right) \pm \sqrt{\frac{9g^2}{4l^2} + \frac{ag}{2l} + g}$$

$$= \frac{g}{2}\left(\frac{3}{l} + \frac{2}{a}\right) \pm \sqrt{\frac{9g^2}{4l^2} + \frac{ag}{2l} + g}$$

合わない

$$\frac{g}{2}\sqrt{\frac{9}{l^2} + \frac{4}{a^2} + \frac{8}{la}}$$

↓

$$\sqrt{\frac{9g^2}{4l^2} + \frac{4g}{4a^2} + \frac{8g^2}{4al}}$$

$$= \sqrt{\frac{9g^2}{4l^2} + \frac{g}{a^2} + \frac{2g^2}{al}}$$

$$(\omega^2-\omega_0^2)\left(\frac{3}{2}\frac{a}{l}\omega^2-\omega_0^2\right)-\frac{a}{l}\omega^4$$

$$=\left(\frac{3}{2}\frac{a}{l}-\frac{a}{l}\right)\omega^4-\omega^2\omega_0^2-\frac{3}{2}\frac{a}{l}\omega^2\omega_0^2+\omega_0^4$$

$$=\frac{1}{2}\frac{a}{l}\omega^4-\omega_0^2\left(\frac{3}{2}\frac{a}{l}+1\right)\omega^2+\omega_0^4=0$$

$$\omega_0^2=\frac{g}{l}$$

$$\frac{1}{2}\frac{a}{l}\omega^4-\frac{g}{l}\left(\frac{3}{2}\frac{a}{l}+1\right)\omega^2+\frac{g^2}{l^2}=0$$

両辺に $\times\frac{2l}{ag}$

$$\frac{1}{g}\omega^4-\frac{2\cancel{l}}{a\cancel{g}}\frac{\cancel{g}}{\cancel{l}}\left(\frac{3}{2}\frac{a}{l}+1\right)\omega^2+\frac{g^{\cancel{2}}}{l^{\cancel{2}}}\times\frac{2\cancel{l}}{a\cancel{g}}=0$$

$$\frac{1}{g}\omega^4-\frac{2}{a}\left(\frac{3}{2}\frac{a}{l}+1\right)\omega^2+\frac{2g}{al}=0$$

$$\frac{1}{g}\omega^4-\left(\frac{3}{l}+\frac{2}{a}\right)\omega^2+\frac{2g}{al}=0$$

$$\omega^2=\frac{1}{\frac{2}{g}}\left\{\left(\frac{3}{l}+\frac{2}{a}\right)\pm\sqrt{\left(\frac{3}{l}+\frac{2}{a}\right)^2-\frac{8}{al}}\right\}$$

$$=\frac{g}{2}\left\{\left(\frac{3}{l}+\frac{2}{a}\right)\pm\sqrt{\frac{9}{l^2}+\frac{12}{al}+\frac{4}{a^2}-\frac{8}{al}}\right\}$$

$$=\frac{g}{2}\left\{\left(\frac{3}{l}+\frac{2}{a}\right)\pm\sqrt{\frac{9}{l^2}+\frac{4}{a^2}+\frac{4}{al}}\right\}$$

$$\therefore\ \omega_+^2=\frac{g}{2}\left\{\left(\frac{3}{l}+\frac{2}{a}\right)+\sqrt{\frac{4}{a^2}+\frac{9}{l^2}+\frac{4}{al}}\right\}$$

$$\omega_-^2=\frac{g}{2}\left\{\left(\frac{3}{l}+\frac{2}{a}\right)-\sqrt{\frac{4}{a^2}+\frac{9}{l^2}+\frac{4}{al}}\right\}$$

(b) $l \gg a$ より

$$\omega^2=\frac{g}{2a}\left\{\left(\frac{3a}{l}+2\right)\pm\sqrt{4+\underbrace{\frac{9a^2}{l^2}}_{0}+\frac{4a}{l}}\right\}$$

$$\fallingdotseq\frac{g}{2a}\left\{\left(\frac{3a}{l}+2\right)\pm 2\left(\frac{a}{2l}+1\right)\right\}$$

$$\boxed{\begin{aligned}\left(4+\frac{4a}{l}\right)^{\frac{1}{2}}&=2\left(1+\frac{a}{l}\right)^{\frac{1}{2}}\\&\approx 2\left(\frac{a}{2l}+1\right)\end{aligned}}$$

$$=\underbrace{\frac{g}{2a}\left(4+\frac{4a}{l}\right)}_{\omega_+^2},\ \underbrace{\frac{g}{2a}\left(\frac{2a}{l}\right)}_{\omega_-^2}$$

$$\omega_+^2=\frac{2g}{a}+\frac{2g}{l}$$

$$\begin{pmatrix}\frac{2g}{a}+\frac{2g}{l}-\frac{g}{l} & \frac{a}{l}\left(\frac{2g}{a}+\frac{2g}{l}\right)\\ \frac{2g}{a}+\frac{2g}{l} & \frac{3}{2}\frac{a}{l}\left(\frac{2g}{a}+\frac{2g}{l}\right)-\frac{g}{l}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0$$

$$\begin{pmatrix}\frac{2g}{a}+\frac{g}{l} & \frac{2g}{l}+\frac{2ag}{l^2}\\ \frac{2g}{a}+\frac{2g}{l} & \frac{2g}{l}+\frac{3ag}{l^2}\end{pmatrix}\begin{pmatrix}A\\B\end{pmatrix}=0$$

$\frac{a}{l} = c$ 　2次c

$\times \frac{a}{g}$

$$\begin{pmatrix} 2+c & 2c+2c^2 \\ 2+2c & 2c+3c^2 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = 0$$

$$(2+c)A + (2c+2c^2)B = 0$$

$$\begin{cases} \frac{A}{B} = -\frac{2c+2c^2}{2+c} \longrightarrow 0 \\ \frac{A}{B} = -\frac{2c+3c^2}{2+2c} \longrightarrow 0 \end{cases}$$

以上とすれば… $c \to 0$ より

$$A \ll B$$