H31 東工大院試 午前

3 [A] $\mathbf{r} = (x, y)$

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$

$\ln|\mathbf{r}|$ の勾配 (grad) を求める

$$\nabla \log|\mathbf{r}| = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) \ln(x^2+y^2)^{\frac{1}{2}}$$

$$= \left(\frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \times 2x, \frac{1}{2}(x^2+y^2)^{-\frac{1}{2}} \times 2y\right)$$

$$= \left(\frac{x}{r^2}, \frac{y}{r^2}\right)$$

$$= \frac{\mathbf{r}}{r^2}$$

又は、

$$\nabla \log r = \frac{\partial}{\partial \mathbf{r}} \log \mathbf{r}$$

r座標だと逆らしい

$$= \frac{\partial r}{\partial \mathbf{r}} \frac{\partial}{\partial r} \log r$$

$$= \mathbf{e}_r \frac{1}{r}$$

$$= \frac{\mathbf{r}}{r^2}$$

$\mathbf{e}_r r = \mathbf{r}$

$\mathbf{e}_r = \frac{\mathbf{r}}{r}$

$\frac{\partial}{\partial r} \log r$

$\frac{\partial \mathbf{r}}{\partial r} \frac{\partial}{\partial r} \log r$

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$\mathbf{e}_r \frac{1}{\mathbf{r}}$

(2) ラプラシアン

$\Delta = \nabla \cdot \nabla$ を $(r\text{-}\theta)$ で表す

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y}$$

$$= \frac{\partial \mathbf{r}}{\partial r} \nabla = \mathbf{e}_r \nabla$$

$$\frac{\partial}{\partial \theta} = \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y}$$

$$= \frac{\partial \mathbf{r}}{\partial \theta} \nabla = r\mathbf{e}_\theta \nabla$$

$(r\cos\theta, r\sin\theta)$

$-r\sin\theta, r\cos\theta$

$r(-\sin\theta, r\cos\theta)$

$= r\mathbf{e}_\theta$

$$\therefore \nabla = \frac{\partial}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \underline{\frac{\partial}{\partial r}\mathbf{e}_r\left(\frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta\right)}$$

$$+ \underline{\frac{1}{r}\frac{\partial}{\partial \theta}\mathbf{e}_\theta\left(\frac{\partial}{\partial r}\mathbf{e}_r\right)} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

$$= \frac{\partial^2}{\partial r^2} + \underline{\frac{1}{r}\frac{\partial}{\partial r}} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

(3) $|\mathbf{r}| \neq 0$ において

$\Delta \ln|\mathbf{r}|$ を求める

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\ln r$$

$$= \frac{\partial}{\partial r}\frac{1}{r} + \frac{1}{r}\frac{1}{r}$$

$$= -\frac{1}{r^2} + \frac{1}{r^2}$$

$$= \underline{0}$$

(4) $|\mathbf{r}| \leq a$ を積分範囲

$$I = \iint \Delta \ln|\mathbf{r}|\, dx\, dy$$

$$= \oint_c dl\, \mathbf{n} \cdot \nabla \log r$$

$$= \oint_c dl\, \mathbf{e}_r \frac{\mathbf{e}_r}{a}$$

$$= 2\pi a \frac{1}{a} = \underline{2\pi}$$

(5) $\Delta \ln|\mathbf{r}|$

$$= \underline{2\pi\delta(\mathbf{r})}$$

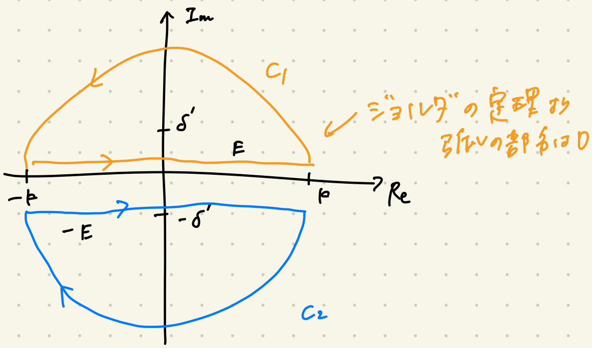
[B](6)　（素粒子でも見たことあるらしい）

$$J = \int_{-\infty}^{\infty} \underbrace{\frac{e^{ipx}}{p^2-k^2-i\delta}}_{f(p)\text{ とおく}}\, dp$$

$$p = \pm\sqrt{k^2+i\delta}$$
$$= \pm k\left(1+\frac{i\delta}{k^2}\right)^{\frac{1}{2}}$$
$$\fallingdotseq \pm k\left(1+\frac{i\delta}{2k^2}\right)$$

$\frac{\delta}{2k^2}$ は無限小なので $\to \delta'$

$$p = \pm k(1+\delta')$$
$$= \pm E \quad \leftarrow \text{極}$$



$$f(p) = \frac{e^{ipx}}{p^2-k^2-i\delta} = \frac{e^{ipx}}{p^2-E^2}$$
$$= \frac{e^{ipx}}{(p+E)(p-E)}$$

$$\frac{e^{ipx}}{(p+E)(p-E)} = e^{ipx}\frac{1}{2E}\left(\frac{1}{(p-E)}-\frac{1}{p+E}\right)$$

i) $x>0$ のとき

$$\oint_{C_1} f(p)\,dp = \int_{-\infty}^{\infty} f(p)\,dp$$
$$= 2\pi i \,\mathrm{Res}\, f(E) = 2\pi i\frac{e^{iEx}}{2E} = \frac{\pi i}{E}e^{iEx}$$

ii) $x<0$　C_2

$$\oint_{C_2} f(p)\,dp$$
$$= \int_{-\infty}^{\infty} f(p)\,dp = -2\pi i\,\mathrm{Res}\, f(-E)$$
$$= \frac{\pi i}{E}e^{-iEx}$$

$$\therefore \int_{-\infty}^{\infty} f(p)\,dp = \frac{\pi i}{E}e^{iE|x|}$$

$$E = k(1+\delta i)$$
$$\delta \to 0$$

$$= \frac{\pi i}{k}e^{ik|x|}$$