1

(1)

$$\mathbf{r} = r(\cos\theta, \sin\theta) = r\,\mathbf{e}_r$$

$$\mathbf{v} = \dot{r}\,\mathbf{e}_r + r\dot{\theta}\,\mathbf{e}_\theta$$

$$\mathbf{a} = \ddot{r}\,\mathbf{e}_r + \dot{r}\dot{\theta}\,\mathbf{e}_\theta + \dot{r}\dot{\theta}\,\mathbf{e}_\theta + r\ddot{\theta}\,\mathbf{e}_\theta + r\dot{\theta}\dot{\theta}(-\mathbf{e}_r)$$

$$\begin{cases} \mathbf{e}_r = (\cos\theta, \sin\theta) \\ \mathbf{e}_\theta = (-\sin\theta, \cos\theta) \end{cases}$$

$$\frac{d}{dt}\mathbf{e}_r = \dot{\theta}\,\mathbf{e}_\theta$$

$$\frac{d}{dt}\mathbf{e}_\theta = \dot{\theta}(-\mathbf{e}_r)$$

$$m\mathbf{a} = -G\frac{Mm}{r^2}\mathbf{e}_r$$

$$m(\ddot{r} - r\dot{\theta}^2) = -G\frac{Mm}{r^2}$$

$$m(2\dot{r}\dot{\theta} + r\ddot{\theta}) = 0 \quad //$$

→ $m\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$ と暗記か-?

(2)

$$m\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$

↓ 積分.

$$m\int \frac{d}{dt}(r^2\dot{\theta})\,dt = 0$$

$$mr^2\dot{\theta} = c$$

$$\underbrace{mr\dot{\theta}}_{p} \times \underbrace{r}_{r} = c$$

角運動量が保存している

(3) $mr^2\dot{\theta} = L$ とする.

$$\dot{\theta} = \frac{L}{mr^2}$$

※動径方向 EOMに代入

$$m\left(\ddot{r} - r\frac{L^2}{m^2r^4}\right) = -G\frac{Mm}{r^2}$$

$$m\ddot{r} = -G\frac{Mm}{r^2} + r\frac{L^2}{m^2r^4}$$

↓ rで積分

$$= \int -G\frac{Mm}{r^2}\,dr + \int \cancel{r}\,\frac{L^2}{m}\frac{1}{r^3}\,dr$$

$$= G\frac{Mm}{r} + -\frac{1}{2}\frac{L^2}{m}\frac{1}{r^2} + C$$

$$m\ddot{r} = -\frac{d}{dr}\underbrace{\left(-G\frac{Mm}{r} + \frac{1}{2}\frac{L^2}{mr^2} + c\right)}_{U}$$

$\lim_{r\to\infty} U = 0$ より

$\lim_{r\to\infty} U = c = 0$

よって

$$U = -G\frac{Mm}{r} + \frac{1}{2}\frac{L^2}{mr^2}$$

(4)

$$m\ddot{r} + \frac{dU}{dr} = 0$$

$$m\ddot{r}\dot{r} + \frac{dr}{dr}\dot{r} = 0$$

$\dot{r}^2$

$$\boxed{\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\frac{dr}{dt}\frac{dr}{dt}\right)\\ &= \frac{1}{2}\left(\ddot{r}\dot{r} + \dot{r}\ddot{r}\right)\\ &= \ddot{r}\dot{r}\end{aligned}}$$

$$m\ddot{r}\dot{r} + \frac{dU}{dr}\dot{r} = 0$$

$$m\int \ddot{r}\dot{r}\,dt + \int \frac{dU}{\cancel{dr}}\frac{\cancel{dr}}{dt}\,\cancel{dt}$$

$$\int \frac{m}{2}\frac{d}{\cancel{dt}}\dot{r}^2 \cancel{dt} + U = E_1$$

$$\frac{1}{2}m\dot{r}^2 + U = E_1$$

エネルギー 角運動量保存，

近日点，遠日点を求める．

↳ 運エが0になる

↳ $\dot{r} = 0$

$$U = E_1$$

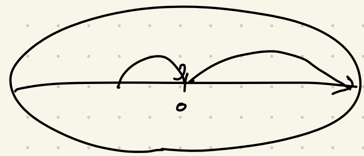
$$\frac{L_0^2}{2mr^2} - \frac{GMm}{r} = E_1 < 0 \quad \leftarrow \text{楕円より}$$

$$E_1 r^2 + GMmr - \frac{L^2}{2m} = 0$$

$$r^2 + \frac{GMm}{E_1}r - \frac{L^2}{2mE_1} = 0$$

$$r = \frac{1}{2}\left\{-\frac{GMm}{E_1} \pm \sqrt{\left(\frac{GMm}{E_1}\right)^2 + 4\frac{L^2}{2mE_1}}\right\}$$

$$= \frac{1}{2}\left\{-\frac{GMm}{E_1} \pm \sqrt{\left(\frac{GMm}{E_1}\right)^2 + \frac{2L^2}{mE_1}}\right\}$$



長軸半径

$$\frac{1}{2}(r^+ - r^-)$$

$$= \frac{1}{2}\left(\frac{1}{2}\left(\bigcirc + \square - \bigcirc + \square\right)\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\left(2\sqrt{\bigcirc^2 + \triangle}\right)\right)$$

$$= \frac{1}{2}\sqrt{\left(\frac{GMm}{E_1}\right)^2 + \left(\frac{2L_1^2}{mE_1}\right)} \quad //$$

(5) 円運動になることから

$r = r^+$ で一定へ．

EOMより

$$\begin{cases} m\left(\ddot{r} - r^+\dot{\theta}^2\right) = -\dfrac{GMm}{r^{+2}} \\ m\dfrac{1}{r^+}\dfrac{d}{dt}\left(r^{+2}\dot{\theta}\right) = 0 \end{cases}$$

r = 一定より

$$\ddot{r} = 0$$

$$\begin{cases} -m r^{*} \dot{\theta}^{2} = -\dfrac{GMm}{r^{*2}} \\ m \dfrac{1}{r^{*}} \dfrac{d}{dt}\left(r^{*2} \dot{\theta}\right) = 0 \end{cases}$$

下式より積分

$$m r^{*} \dot{\theta} = C$$

$$r^{*} \dot{\theta} = C$$

$$\boldsymbol{r} = r\boldsymbol{e}_r$$

$$\boldsymbol{v} = \dot{r}\boldsymbol{e}_r + r\dot{\theta}\boldsymbol{e}_\theta$$

より

$$v_\theta = r\dot{\theta}$$

$$v = C$$

上式に代入

$$m \frac{v^2}{r^{*}} = \frac{GMm}{r^{*2}}$$

$$\frac{1}{2} m v^2 = \frac{GMm}{2r^{*}}$$

$$E_c = \frac{GMm}{2r^{*}} + \left(-\frac{GMm}{r^{*}}\right)$$