2020 東工大理学院 量子力学

[A] (1)

$$[\hat{\mathcal{H}}, \hat{a}^\dagger] = [\hbar\omega(\hat{a}^\dagger\hat{a}+\tfrac{1}{2}), \hat{a}^\dagger]$$
$$= \hbar\omega[a^\dagger a, a^\dagger]$$
$$= \hbar\omega\left(a^\dagger[a, a^\dagger] + [a^\dagger, a^\dagger]a\right)$$
$$= \underline{\hbar\omega a^\dagger} \ \text{(う)} \quad \dashrightarrow [\mathcal{H}, \hat{a}] = \underline{-\hbar\omega\hat{a}} \ \text{(え)}$$

(2) $a_H^\dagger$, a_H をハイゼンベルグ描像での生成・消滅演算子

$$\hat{O}_H(t) = \exp\left(\tfrac{i}{\hbar}\hat{\mathcal{H}}t\right)\hat{O}_s\exp\left(-\tfrac{i}{\hbar}\hat{\mathcal{H}}t\right)$$

$$\begin{cases} \hat{a}_H^\dagger = \exp\left(\tfrac{i}{\hbar}\hat{\mathcal{H}}t\right)\hat{a}^\dagger\exp\left(-\tfrac{i}{\hbar}\hat{\mathcal{H}}t\right) \\ \hat{a}_H = \exp\left(\tfrac{i}{\hbar}\hat{\mathcal{H}}t\right)\hat{a}\exp\left(-\tfrac{i}{\hbar}\hat{\mathcal{H}}t\right) \end{cases}$$

$$\frac{d}{dt}\hat{a}_H^\dagger = \frac{i}{\hbar}\mathcal{H}a_H^\dagger - \frac{i}{\hbar}a_H^\dagger\mathcal{H}$$
$$= \frac{i}{\hbar}[\mathcal{H}, \hat{a}_H^\dagger] = \frac{i}{\hbar}\hbar\omega\hat{a}_H^\dagger = \underline{i\omega\hat{a}_H^\dagger} \ \text{(お)}$$

$$\frac{d}{dt}\hat{a}_H = \frac{i}{\hbar}[\mathcal{H}, \hat{a}_H] = \underline{-i\omega\hat{a}_H} \ \text{(か)}$$

$\frac{d}{dt}\hat{a}_H^\dagger = i\omega\hat{a}_H^\dagger$ より $\hat{a}_H = Ae^{i\omega t}$

初期条件より $\hat{a}_H(0) = \hat{a}^\dagger$ $\therefore \underline{\hat{a}_H^\dagger(t) = \hat{a}^\dagger e^{i\omega t}}$ (き)

同様に

$\frac{d}{dt}\hat{a}_H(t) = -i\omega\hat{a}_H(t)$ $\therefore \underline{\hat{a}_H(t) = \hat{a}e^{-i\omega t}}$ (く)

最初の定義より

$$\hat{a}_H^\dagger(t) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x}_H(t) - \frac{i}{\sqrt{m\omega}}\hat{p}_H(t)\right)$$
$$+)\quad \hat{a}_H(t) = \frac{1}{\sqrt{2\hbar}}\left(\sqrt{m\omega}\hat{x}_H(t) + \frac{i}{\sqrt{m\omega}}\hat{p}_H(t)\right)$$
$$\cdots \quad \underline{\hat{x}_H(t) = \frac{1}{2}\sqrt{\frac{2\hbar}{m\omega}}\left(\hat{a}_H^\dagger + \hat{a}_H\right)}$$

シュレディンガー描像に書き直して、

$$\underline{\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger e^{i\omega t} + \hat{a}e^{-i\omega t}\right)} \ \text{(け)(こ)}$$

[B]

(3) $L = \frac{m}{2}\dot{x}^2 - \frac{m\omega^2}{2}x^2 + F(t)x$, $F(t) = F_0\delta(t)$

ラグランジアンからハミルトニアンを求めるには、ルジャンドル変換。

$$\boxed{\mathcal{H} = \sum_i p_i\dot{x}_i - L}$$

1粒子

$$= m\dot{x}\dot{x} - \frac{m}{2}\dot{x}^2 + \frac{m\omega^2}{2}x^2 - F(t)x$$
$$= \frac{m}{2}\dot{x}^2 + \frac{m\omega^2}{2}x^2 - F(t)x$$

ハイゼンベルグ方程式

$$i\hbar\frac{d}{dt}\hat{A}_H = [\hat{A}_H, \hat{\mathcal{H}}]$$

Date

[A ハイゼンベルグ方程式より

$$\frac{d}{dt}\hat{x}_H = \frac{1}{i\hbar}[\hat{x}_H, \mathcal{H}]$$

$$= \frac{1}{i\hbar}\left[\hat{x}_H, \frac{\hat{p}_H^2}{2m}\right] = \frac{1}{i\hbar}\times\frac{1}{2m}\times 2i\hbar\,\hat{p}_H = \frac{\hat{p}_H(t)}{m}$$

求めたいのは

$$\frac{d^2}{dt^2}\hat{x}_H = \frac{1}{m}\frac{d}{dt}\hat{p}_H(t)$$

$$\left(\begin{aligned}\frac{d}{dt}\hat{p}_H &= \frac{1}{i\hbar}[\hat{p}_H, \mathcal{H}]\\ &= \frac{1}{i\hbar}\left\{\frac{m\omega^2}{2}[\hat{p}_H, \hat{x}_H^2] - F(t)[\hat{p}_H, \hat{x}_H]\right\}\\ &= F(t) - m\omega^2\hat{x}_H\end{aligned}\right)$$

$$\frac{d^2}{dt^2}\hat{x}_H = \frac{F(t)}{m} - \omega^2\hat{x}_H \quad \text{(エ)}$$

(4)

- $\lim_{\varepsilon\to+0}\left(\hat{x}_H(\varepsilon) - \hat{x}_H(-\varepsilon)\right) = 0$ (し)

- $\lim_{\varepsilon\to+0}\left(\frac{d}{dt}\hat{x}(\varepsilon) - \frac{d}{dt}\hat{x}(-\varepsilon)\right)$

$$= \int_{-\varepsilon}^{\varepsilon}\frac{d^2}{dt^2}\hat{x}_H(t)\,dt = \lim_{\varepsilon\to0}\int_{-\varepsilon}^{\varepsilon}\frac{F(t)}{m} - \omega^2\hat{x}_H\,dt = \frac{F_0}{m} \quad \text{(す)}$$

Date

(5) $t<0$ 撃力を加える前

$$\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}^\dagger e^{i\omega t} + \hat{a}e^{-i\omega t}\right)$$

$t>0$ では.

$$\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{b}^\dagger e^{i\omega t} + \hat{b}e^{-i\omega t}\right)$$ とおく.

(4)の条件を用いて

① $\lim_{\varepsilon\to0}\left(\hat{x}_H(\varepsilon) - \hat{x}_H(-\varepsilon)\right) = \sqrt{\frac{\hbar}{2m\omega}}\left(b^\dagger + b - a^\dagger - a\right) = 0$

$$\therefore\ b^\dagger + b = a^\dagger + a \quad \cdots ①'$$

② $\lim_{\varepsilon\to0}\left(\frac{d}{dt}\hat{x}_H(\varepsilon) - \frac{d}{dt}\hat{x}_H(-\varepsilon)\right)$

$$\therefore\ b^\dagger - b = a^\dagger - a - F_0 i\sqrt{\frac{2}{m\omega\hbar}} \quad \cdots ②'$$

①'②'を連立して

$$\hat{b}^\dagger = \hat{a}^\dagger - iF_0\sqrt{\frac{1}{2m\omega\hbar}} \quad \text{(せ)}$$

$$\hat{b} = \hat{a} + iF_0\sqrt{\frac{1}{2m\omega\hbar}} \quad \text{(そ)}$$

(6)

- $\hat{x}_H(t) = \sqrt{\frac{\hbar}{2m\omega}}\left\{\left(\hat{a}^\dagger - iF_0\sqrt{\frac{1}{2m\omega\hbar}}\right)e^{i\omega t} + \left(\hat{a} + iF_0\sqrt{\frac{1}{2m\omega\hbar}}\right)e^{-i\omega t}\right\}$

- $\hat{x}_H(t)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\hat{a}^\dagger|0\rangle + iF_0\sqrt{\frac{1}{2m\hbar\omega}}\left(e^{-i\omega t} - e^{i\omega t}\right)|0\rangle\times\sqrt{\frac{\hbar}{2m\omega}}$

- $\langle 0|\hat{x}_H(t)|0\rangle = \sqrt{\frac{\hbar}{2m\omega}}\underbrace{\langle 0|a^\dagger|0\rangle}_{=0} + iF_0\sqrt{\frac{1}{2m\hbar\omega}}\left(e^{-i\omega t} - e^{i\omega t}\right)\times\sqrt{\frac{\hbar}{2m\omega}}$

$$= \frac{F_0}{m\omega}\sin\omega t \quad \text{(た)}$$

Date

$$\circ \ \langle 0|\hat{x}_H^2(\tau)|0\rangle = \langle 0|\left(\sqrt{\frac{\hbar}{2m\omega}}\,(\overset{e^-}{a} + \overset{e^+}{a^\dagger}) + A\right)^2|0\rangle$$

$$= \langle 0|\frac{\hbar}{2m\omega}(\overset{e^-}{a}+\overset{e^+}{a^\dagger})^2 + 2A\sqrt{\frac{\hbar}{2m\omega}}(a+a^\dagger) + A^2|0\rangle$$

$$= \langle 0|\frac{\hbar}{2m\omega}(\overset{e^{2+}}{\hat{a}^2} + aa^\dagger + a^\dagger a + \overset{e^{2-}}{a^{\dagger 2}}) + A^2|0\rangle$$

$$= \langle 0|\frac{\hbar}{2m\omega}aa^\dagger + A^2|0\rangle$$

$$= \frac{\hbar}{2m\omega} + A^2$$

$$\circ \ \langle 0|\hat{x}_H^2(\tau)|0\rangle - \langle 0|\hat{x}_H(\tau)|0\rangle^2 = \underline{\underline{\frac{\hbar}{2m\omega}}} \quad (5)$$