初めてでも分かる

数 III 積分演習

Ver.1.28

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11 February, 2024

0.1 有理式の積分

1 ∗∗ $\displaystyle\int \frac{x^3}{(x+1)(x+2)}dx$

−有理式 $\left(\frac{\text{多項式}}{\text{多項式}}\right)$ の積分−

1. 分子の次数を分母の次数より下げる
2. $\frac{f'(x)}{f(x)}$ の形を見出す
3. 部分分数分解する
4. $\int \frac{1}{x^2+a^2}dx$ に帰着させる

2 ∗∗ $\displaystyle\int_0^{2\pi} \sqrt{1+\cos x}dx$

根号は基本的には外す

1 $\displaystyle I = \int \frac{x^3}{(x+1)(x+2)}dx$

$x^3 = (x^2+3x+2)(x-3)+7x+6$ より

$$\frac{x^3}{(x+1)(x+2)} = x-3+\frac{7x+6}{(x+1)(x+2)}$$ から

$$\therefore I = \int \left(x-3+\frac{-1}{x+1}+\frac{8}{x+2}\right)dx$$
$$= \frac{x^2}{2}-3x-\log|x+1|+8\log|x+2|+C$$ ■

2 $\displaystyle\int_0^{2\pi} \sqrt{1+\cos x}dx$

$$\begin{aligned}\sqrt{1+\cos x} &= \sqrt{2\cos^2\frac{x}{2}}\\ &= \sqrt{2}|\cos\frac{x}{2}|\\ &= \begin{cases}\sqrt{2}\cos\frac{x}{2} & (0 \leq x \leq \pi)\\ -\sqrt{2}\cos\frac{x}{2} & (\pi \leq x \leq 2\pi)\end{cases}\end{aligned}$$

$$\begin{aligned}\int_0^{2\pi}\sqrt{1+\cos x}dx &= \int_0^{\pi}\sqrt{1+\cos x}dx + \int_{\pi}^{2\pi} -\sqrt{1+\cos x}dx\\ &= \sqrt{2}\left\{\left[2\sin\frac{x}{2}\right]_0^{\pi} - \left[2\sin\frac{x}{2}\right]_{\pi}^{2\pi}\right\}\\ &= \sqrt{2}(2-0)-(0-2)\\ &= 4\sqrt{2}\end{aligned}$$ ■

$\boxed{3}$ ＊＊＊＊ $\displaystyle\int_0^1 \sqrt{\frac{1-x}{1+x}}dx$

$\sqrt{\frac{ax+b}{cx+d}}$ 型は $t=\sqrt{\frac{ax+b}{cx+d}}$ を置換して
t だけの式を作る

$\boxed{3}$ $\displaystyle\int_0^1 \sqrt{\frac{1-x}{1+x}}dx$

$$t^2=\frac{1-x}{1+x}$$
$$x(1+x^2)=1-t^2$$
$$x=\frac{1-t^2}{1+t^2}$$
$$\frac{dx}{dt}=-\frac{4t^2}{1+t^2}$$

$$t=\sqrt{\frac{1-x}{1+x}}$$

x	0	$\to$	1
t	1	$\to$	0

$$\frac{dx}{dt}=-\frac{4t^2}{1+t^2}$$

$$\int_0^1 \sqrt{\frac{1-x}{1+x}}dx=\int_1^0 t\left\{-\frac{4t}{(1+t^2)^2}dt\right\}$$
$$=4\int_0^1 \frac{t^2}{(1+t^2)^2}dt$$

$$=4\int_0^{\frac{\pi}{4}} \tan^2\theta\cdot\cos^4\theta$$
$$\cdot\frac{1}{\cos^2\theta}d\theta$$
$$=\int_0^{\frac{\pi}{4}}\sin^2\theta d\theta$$
$$=\int_0^{\frac{\pi}{4}}\frac{1-\cos 2\theta}{2}d\theta$$
$$=\frac{1}{2}\left[\theta-\frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$$
$$=\frac{\pi}{8}-\frac{1}{4} \quad \blacksquare$$

$$t=\tan\theta$$

t	0	$\to$	1
θ	0	$\to$	$\frac{\pi}{4}$

$$dt=\frac{1}{\cos^2\theta}d\theta$$

4 ∗ $\displaystyle\int x\sqrt{x+1}dx$

ルートの中身と同じ形を作る

5 ∗ $\displaystyle\int \frac{1}{x^2}(1+\frac{2}{x})^2dx$

微分系の接触を利用

6 ∗∗ $\displaystyle\int_0^1 \sqrt{1+2\sqrt{x}}dx$

かたまりを置換する

4 $\displaystyle\int x\sqrt{x+1}dx$

$$\begin{aligned} x\sqrt{x+1} &= (x+1)\sqrt{x+1}-\sqrt{x+1} \\ &= (x+1)^{\frac{3}{2}}-(x+1)^{\frac{1}{2}} \\ \int x\sqrt{x+1}dx &= \int (x+1)^{\frac{3}{2}}-(x+1)^{\frac{1}{2}}dx \\ &= \frac{2}{5}(x+1)^{\frac{5}{2}}-\frac{2}{3}(x+1)^{\frac{3}{2}} \end{aligned}$$ ■

5 $\displaystyle\int \frac{1}{x^2}(1+\frac{2}{x})^2dx$

$$\left\{1+\frac{2}{x}\right\}'=-\frac{2}{x^2}\text{より}$$

$$\int -\frac{1}{2}(1+\frac{2}{x})'(1+\frac{2}{x})^2dx=-\frac{1}{6}(1+\frac{2}{x})^3+C$$ ■

6 $\displaystyle\int_0^1 \sqrt{1+2\sqrt{x}}dx$

$t=\sqrt{1+2\sqrt{x}}$ と置換

$$\begin{aligned} x &= (\frac{t^2-1}{2})^2 \\ &= \frac{1}{4}(t^2-1)^2 \end{aligned}$$

$x=\frac{1}{4}(t^2-1)^2$

x	0	$\to$	1
t	1	$\to$	$\sqrt{3}$

$dx=t(t^2-1)dt$

$$\begin{aligned} \int_0^1 \sqrt{1+2\sqrt{x}}dx &= \int_1^{\sqrt{3}} t\cdot t(t^2-1)dt \\ &= \left[\frac{1}{5}t^5-\frac{1}{3}t^3\right]_1^{\sqrt{3}} \\ &= \frac{4}{5}\sqrt{3}+\frac{2}{15} \end{aligned}$$ ■

7 $*$ $\displaystyle\int_1^2 \frac{1}{2^x}dx$

対数微分法を利用

7

$$(2^{-x})' = -2^{-x}\log 2$$
$$2^{-x} = (-\frac{2^{-x}}{\log 2})'$$

対数微分法

$$y = 2^{-x}$$
$$\log y = -x\log 2$$
$$\frac{y'}{y} = -\log 2$$
$$y' = -2^{-x}\log 2$$

$$\int_1^2 \frac{1}{2^x}dx = \left[-\frac{2^{-x}}{\log 2}\right]_1^2 = -\frac{1}{\log 2}(2^{-2} - 2^{-1})$$
$$= \frac{1}{4\log 2} \quad \blacksquare$$

8 $*$ $\displaystyle\int_0^1 \frac{1}{x^2-2x+4}$

$\frac{1}{x^2+a^2}$ の形に帰着する

8 $\displaystyle\int_0^1 \frac{1}{x^2-2x+4} = \int_0^1 \frac{1}{(x-1)^2+3}$

$$x - 1 = \sqrt{3}\tan\theta$$
$$dx = \sqrt{3}\frac{1}{\cos^2\theta}d\theta$$

x	0	$\to$	1
θ	$-\frac{\pi}{6}$	$\to$	0

$$= \int_{-\frac{\pi}{6}}^0 \frac{\cos^2\theta}{3}\cdot\sqrt{3}\cdot\frac{1}{\cos^2\theta}d\theta$$
$$= \frac{\sqrt{3}}{3}\cdot\frac{\pi}{6}$$
$$= \frac{\sqrt{3}}{18}\pi \quad \blacksquare$$

9 ****　$\displaystyle\int \frac{1}{\sqrt{1+x^2}}dx$

$\sqrt{1+x^2}$ を含む形
$\rightarrow x=\frac{e^t-e^{-t}}{2}(=\sinh x)$ と置換

9

$$2x = e^t - e^{-t}$$
$$(e^t)^2 - 2xe^t - 1 = 0$$
$$e^t = x + \sqrt{x^2+1}$$

$$x = \frac{e^t-e^{-t}}{2}$$
$$t = \log\left(x+\sqrt{x^2+1}\right)$$
$$dx = \frac{e^t+e^{-t}}{2}dt$$

$$\begin{aligned}\int \frac{1}{\sqrt{1+x^2}}dx &= \int \frac{1}{\sqrt{1+\frac{(e^t-e^{-t})^2}{4}}}\frac{e^t+e^{-t}}{2}dt\\ &= \int \frac{1}{\frac{e^t+e^{-t}}{2}}\cdot\frac{e^t+e^{-t}}{2}dt\\ &= \int dt\\ &= t+C\\ &= \log\left(x+\sqrt{x^2+1}\right)+C \quad \blacksquare\end{aligned}$$

10 **　$\displaystyle\int x\sqrt{3x-5}dx$

和を含めて根号は基本的には外す

10

$$\int \frac{t^2+5}{3}\cdot t\cdot\frac{2}{3}tdt$$
$$=\frac{2}{9}\int t^4+5t^2dt$$
$$=\frac{2}{9}\left(\frac{1}{5}t^5+\frac{5}{3}t^3\right)+C$$

$$t=\sqrt{3x-5}$$
$$x=\frac{t^2+5}{3}$$
$$dx=\frac{2}{3}tdt$$

$$=\frac{2}{45}(3x-5)\sqrt{3x-5}(3x+\frac{10}{3})+C \quad \blacksquare$$

11 ＊　$\displaystyle\int_0^2 \frac{x}{4+x^2}dx$

置換して整理する

12 ＊＊＊　$\displaystyle\int \frac{x}{(x^2+2)(x^2+3)}dx$

分母が因数分解
→ 部分分数分解

13 ＊　$\displaystyle\int_{-1}^1 \frac{x^3}{1+x^2}dx$

積分区間が対称
→ 偶奇に注目する

14 ＊＊＊　$\displaystyle\int \frac{4(3+3x-x^2)}{(x-1)^2(x+1)}dx$

分母が因数分解
→ 部分分数分解

11　$\displaystyle\int_0^2 \frac{x}{4+x^2}dx$

$u = 4+x^2,\quad du = 2xdx$ と置換

$$\begin{aligned} I &= \int_4^5 \frac{\frac{1}{2}du}{u} \\ &= \left[\frac{1}{2}\log|u|\right]_4^5 \\ &= \frac{1}{2}(\log 5 - \log 4) \quad \blacksquare \end{aligned}$$

12

$$\begin{aligned} \int \frac{x}{(x^2+2)(x^2+3)}dx &= \int x\left(\frac{1}{x^2+2} - \frac{1}{x^2+3}\right)dx \\ &= \int \left(\frac{x}{x^2+2} - \frac{x}{x^2+3}\right)dx \\ &= \int \left\{\frac{1}{2}\cdot\frac{(x^2+2)'}{x^2+2} - \frac{1}{2}\frac{(x^2+3)'}{x^2+3}\right\}dx \\ &= \frac{1}{2}\log\frac{x^2+2}{x^2+3} + C \quad \blacksquare \end{aligned}$$

13　$\dfrac{x^3}{1+x^2}$ は奇関数である為,

$$\int_{-1}^1 \frac{x^3}{1+x^2}dx = 0 \quad \blacksquare$$

14

$$\begin{aligned} &\int \frac{4(3+3x-x^2)}{(x-1)^2(x+1)}dx \\ &= \int \left\{-\frac{3}{x-1} + \frac{10}{(x-1)^2} - \frac{1}{x+1}\right\}dx \\ &= \log\left|\frac{1}{(x-1)^3(x+1)}\right| - \frac{10}{x-1} + C \quad \blacksquare \end{aligned}$$

15 ∗∗∗∗ $\displaystyle\int_0^1 \sqrt{\frac{x}{1+x}}dx$

$\frac{1}{1+A}$ の形
→ を $\tan^2\theta$ にする

15

$x=\tan^2\theta$

$dx=\dfrac{2\tan x}{\cos^2\theta}d\theta$

x	0	→	1
θ	0	→	$\frac{\pi}{4}$

$t=\sin\theta$

$dt=\cos\theta d\theta$

x	0	→	1
θ	0	→	$\frac{\pi}{4}$

$$
\begin{aligned}
&\int_0^1 \sqrt{\frac{x}{1+x}}dx\\
&=\int_0^{\frac{\pi}{4}}\sqrt{\frac{\tan^2\theta}{1+\tan^2\theta}}\cdot 2\tan\theta\frac{1}{\cos^2\theta}d\theta\\
&=\int_0^{\frac{\pi}{4}} 2\tan\theta\cdot\cos\theta\cdot\frac{\sin\theta}{\cos^3\theta}d\theta\\
&=\int_0^{\frac{\pi}{4}}\frac{2\sin^2\theta}{\cos^3\theta}d\theta\\
&=\int_0^{\frac{\pi}{4}}\sin\theta\cdot\left(\frac{1}{\cos^2\theta}\right)'d\theta\\
&=\left[\sin\theta\cdot\frac{1}{\cos^2\theta}\right]^{\frac{\pi}{4}}{}_0-\int_0^{\frac{\pi}{4}}\frac{1}{\cos^2\theta}d\theta\\
&=\sqrt{2}-\int_0^{\frac{\pi}{4}}\frac{\cos\theta}{\cos^2\theta}d\theta\\
&=\sqrt{2}-\int_0^{\frac{\pi}{4}}\frac{\cos\theta}{1-\sin^2\theta}d\theta\\
&=\sqrt{2}-\int_0^{\frac{1}{\sqrt{2}}}\frac{1}{1-t^2}dt\\
&=\sqrt{2}-\int_0^{\frac{1}{\sqrt{2}}}\left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt\\
&=\sqrt{2}-\frac{1}{2}\left[\log|1+t|-\log|1-t|\right]_0^{\frac{1}{\sqrt{2}}}\\
&=\sqrt{2}-\frac{1}{2}\log\frac{\sqrt{2}+1}{\sqrt{2}-1}\\
&=\sqrt{2}-\frac{1}{2}\log\left(\sqrt{2}+1\right)^2\\
&=\sqrt{2}-\log\left(\sqrt{2}+1\right) \quad ■
\end{aligned}
$$

16 ∗∗∗ $\displaystyle\int\sqrt{x\sqrt{x\sqrt{x\cdots}}}dx$

n 乗根 → $\frac{1}{n}$ 乗

16

$$
\begin{aligned}
\sqrt{x\sqrt{x\sqrt{x\cdots}}}&=(x(x(x\cdots)^{\frac{1}{2}})^{\frac{1}{2}})^{\frac{1}{2}}\\
&=x^{\frac{1}{2}}x^{\frac{1}{4}}x^{\frac{1}{8}}\cdots\\
&=x^{\sum_{n=1}^{\infty}\frac{1}{2^n}}=x
\end{aligned}
$$

$$\therefore\int\sqrt{x\sqrt{x\sqrt{x\cdots}}}dx=\int xdx=\frac{1}{2}x^2+C \quad ■$$

17 ＊＊＊　$\displaystyle\int_{\frac{1}{2}}^{1} x\sqrt{\frac{1}{x}-1}dx$

$\sqrt{a^2-x^2}$ を含む形
$x = a\sin\theta$ と置換

17

$$x-\frac{1}{2}=\frac{1}{2}\sin\theta$$
$$dx=\frac{1}{2}\cos\theta d\theta$$

x	$\frac{1}{2}$	$\to$	1
θ	0	$\to$	$\frac{\pi}{2}$

$$\begin{aligned}
&\int_{\frac{1}{2}}^{1} x\sqrt{\frac{1}{x}-1}dx\\
&=\int_{\frac{1}{2}}^{1}\sqrt{x-x^2}dx\\
&=\int_{\frac{1}{2}}^{1}\sqrt{\frac{1}{4}-(x-\frac{1}{2})^2}dx\\
&=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sqrt{1-\sin^2\theta}\cdot\frac{1}{2}\cos\theta d\theta\\
&=\frac{1}{4}\int_0^{\frac{\pi}{2}}\cos^2\theta d\theta\\
&=\frac{1}{4}\int_0^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta\\
&=\frac{1}{4}\left[\frac{\theta}{2}+\frac{\sin 2\theta}{4}\right]_0^{\frac{\pi}{2}}\\
&=\frac{\pi}{16}\quad\blacksquare
\end{aligned}$$

18 ＊＊　$\displaystyle\int\frac{1}{x(x+1)(x+2)}dx$

分母が因数分解
→ 部分分数分解

18

$$\begin{aligned}
&\int\frac{1}{x(x+1)(x+2)}dx\\
&=\int\left(\frac{1}{2x}-\frac{1}{x+1}+\frac{1}{2(x+2)}\right)dx\\
&=\frac{1}{2}\log|x|-\log|x+1|+\frac{1}{2}\log|x+2|+C\quad\blacksquare
\end{aligned}$$

19 ＊＊　$\displaystyle\int\frac{x}{1-x^2}dx$

$\frac{1\text{ 次式}}{2\text{ 次式}}$ → 分子に分母の微分系をつくる

19

$$\begin{aligned}
\int\frac{x}{1-x^2}dx&=-\frac{1}{2}\int\frac{(1-x^2)'}{1-x^2}dx\\
&=-\frac{1}{2}\log|1-x^2|+C\quad\blacksquare
\end{aligned}$$

20 ∗∗∗ $\displaystyle\int_{\sqrt{2}}^{2}\frac{1}{x\sqrt{x^2-1}}dx$

根号 → 基本的には外す

20

$$t=\sqrt{x^2-1}$$
$$dt=\frac{x}{\sqrt{x^2-1}}dx$$

x	$\sqrt{2}$	$\to 2$
t	1	$\to\sqrt{3}$

$$t=\tan\theta$$
$$dt=\frac{1}{\cos^2\theta}d\theta$$

t	1	$\to$	$\sqrt{3}$
θ	$\frac{\pi}{4}$	$\to$	$\frac{\pi}{3}$

$$\int_{\sqrt{2}}^{2}\frac{1}{x\sqrt{x^2-1}}dx$$
$$=\int_{1}^{\sqrt{3}}\frac{1}{t^2+1}dt$$
$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta$$
$$=\frac{\pi}{12}\quad\blacksquare$$

21 ∗∗ $\displaystyle\int\frac{1}{x\sqrt{1-x^2}}dx$

根号 → 基本的には外す

21

$$t=\sqrt{1-x^2}$$
$$dt=-\frac{x}{\sqrt{1-x^2}}dx$$

$$\int\frac{1}{x\sqrt{1-x^2}}dx$$
$$=\int\frac{1}{t^2-1}dt$$
$$=\frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt$$
$$=\frac{1}{2}\log|t-1|-\frac{1}{2}\log|t+1|+C$$
$$=\frac{1}{2}\log|\sqrt{1-x^2}-1|$$
$$-\frac{1}{2}\log|\sqrt{1-x^2}+1|+C\quad\blacksquare$$

22 ∗ $\displaystyle\int_0^{\pi} |\cos x - \cos 2x|\, dx$

絶対値の中身の正負で場合分け

22

$f(x) = \cos x - \cos 2x$ とおいて符号を調べる

$$
\begin{aligned}
&= \cos x - (2\cos^2 x - 1)\\
&= -2\cos^2 x + \cos x + 1\\
&= (1-\cos x)(2\cos x + 1)
\end{aligned}
$$

積分区間である $0 \leq x \leq \pi$の時, $1 - \cos x \geq 0$ が成立するので

$2\cos x + 1 = 0$ となる $x = \dfrac{2}{3}\pi$の前後で正から負に変わる

$$
\begin{aligned}
\int_0^{\pi} |f(x)|\, dx &= \int_0^{\frac{2}{3}\pi} f(x)dx + \int_{\frac{2}{3}\pi}^{\pi} -f(x)dx\\
&= \int_0^{\frac{2}{3}\pi} f(x)dx + \int_{\pi}^{\frac{2}{3}\pi} f(x)dx\\
&= \left[\sin x - \frac{1}{2}\sin 2x\right]_0^{\frac{2}{3}\pi} + \left[\sin x - \frac{1}{2}\sin 2x\right]_{\pi}^{\frac{2}{3}\pi}\\
&= 2\left\{\frac{\sqrt{3}}{2} - \frac{1}{2}\left(-\frac{\sqrt{3}}{2}\right)\right\}\\
&= \frac{3}{2}\sqrt{3} \qquad \blacksquare
\end{aligned}
$$

0.2　指数/対数関数の積分

23 ∗∗ $\displaystyle\int_1^{e} \sqrt{x}\log x dx$

$e^x, \sin x, x^{\alpha}, \log x$

$\longleftarrow$

積分の優先度

これらの積 → 部分積分

23

$$
\begin{aligned}
\int_1^{e} x^{\frac{1}{2}}\log x dx &= \left[\frac{2}{3}x^{\frac{3}{2}}\log x\right] - \int_1^{e} \frac{2}{3}x^{\frac{3}{2}} \cdot x^{-1}dx\\
&= \frac{2}{3}e^{\frac{3}{2}} - \frac{2}{3}\int_1^{e} x^{\frac{1}{2}}dx\\
&= \frac{2}{9}e\sqrt{e} + \frac{4}{9} \qquad \blacksquare
\end{aligned}
$$

24 $*$ $\displaystyle\int_0^2 \frac{3x^3+12x+1}{x^2+4}dx$

(分子の次数)≥(分母の次数)
⟶ 整数の剰余で次数下げ

24 $\displaystyle\int_0^2 \frac{3x^3+12x+1}{x^2+4}dx$

分母と同じ形を作って, 和の形をつくる.

$x = 2\tan\theta$

$dx = 2\dfrac{1}{\cos^2\theta}d\theta$

x	$0 \rightarrow 2$
θ	$0 \rightarrow \frac{\pi}{4}$

$$\begin{aligned}
&\int_0^2 \frac{3x^2+12x+1}{x^2+4}dx \\
&= \int_0^2 \frac{3x(x^2+4)+1}{x^2+4}dx \\
&= 3\int_0^2 xdx + \int_0^2 \frac{1}{x^2+4}dx \\
&= 3\left[\frac{x^2}{2}\right]_0^2 + \int_0^{\frac{\pi}{4}} \frac{1}{4(\tan^2\theta+1)}\frac{2d\theta}{\cos^2\theta} \\
&= 6 + \frac{1}{2}\int_0^{\frac{\pi}{4}} d\theta \\
&= 6 + \frac{\pi}{8} \quad \blacksquare
\end{aligned}$$

25 $**$ $\displaystyle\int (\log x)^2\,dx$

$\log x \rightarrow$ 部分積分
or
$t = \log x$

25

$$\begin{aligned}
\int (\log x)^2\,dx &= \int 1\cdot(\log x)^2\,dx \\
&= x(\log x)^2 - \int x\cdot 2\log x\cdot\frac{1}{x}dx \\
&= x(\log x)^2 - 2\int 1\cdot\log x dx \\
&= x(\log x)^2 - 2\left(x\log x - \int x\cdot\frac{1}{x}dx\right) \\
&= x(\log x)^2 - 2(x\log x - x) + C \quad \blacksquare
\end{aligned}$$

26 $**$ $\displaystyle\int \frac{1}{x(1+\log x)}dx$

$\log x \rightarrow$ ~~部分積分~~
$\rightarrow t = \log x$ と置換
$\rightarrow$ 指数関数へ

26 $\displaystyle\int \frac{1}{x(1+\log x)}dx$

$t = \log x$

$dt = \dfrac{1}{x}$

$dx = xdt$

$$\begin{aligned}
&= \int \frac{1}{1+t}dt \\
&= \log|1+t| + C \\
&= \log|1+\log x| + C \quad \blacksquare
\end{aligned}$$

27 ∗∗ $\displaystyle\int \frac{x}{\cos^2 x}dx$

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$(\tan x)' = \frac{1}{\cos^2 x}$
を部分積分のレパートリーに

28 ∗ $\displaystyle\int \log_2 x dx$

底が e 以外の対数
底を e に変換

29 ∗∗∗ $\displaystyle\int \tan x \log\left(\cos^2 x\right) dx$

対数関数の中身が複雑
→ 丸ごと置換

27
$$\begin{aligned}\int \frac{x}{\cos^2 x}dx &= \int x(\tan x)' \, dx \\ &= x\tan x - \int \tan x dx \\ &= x\tan x + \int \frac{(\cos x)'}{\cos x}dx \\ &= x\tan x + \log|\cos x| + C\end{aligned}$$ ■

28
$$\begin{aligned}\int \log_2 x dx &= \int \frac{\log x}{\log 2} \\ &= \frac{1}{\log 2}\int \log x dx \\ &= \frac{1}{\log 2}\left(x\log x - \int x\cdot\frac{1}{x}dx\right) \\ &= \frac{1}{\log 2}(x\log x - x) + C\end{aligned}$$ ■

29

$t = \cos x$
$dt = -\sin x dx$

$$\begin{aligned}&\int \tan x \log\left(\cos^2 x\right) dx \\ &= \int \frac{\sin x}{\cos x}\log|\cos x|^2 dx \\ &= -\int \frac{1}{t}\log|t|^2 dt \\ &= -2\int (\log|t|)' \log|t| dt \\ &= -2\cdot\frac{1}{2}(\log|t|)^2 + C \\ &= -(\log|\cos x|)^2 + C\end{aligned}$$ ■

30 ∗∗∗ $\displaystyle\int 2^{\log x}dx$

$a^{\log_b x} = x^{\log_b a}$

31 ∗∗ $\displaystyle\int \frac{\log(\log x)}{x\log x}dx$

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対数関数の中身が複雑
→ 丸ごと置換

32 ∗∗ $\displaystyle\int \frac{1}{x(\log x)^2}dx$

$\log x$ が入った積分に $\frac{1}{x}$ があるとき
→ 微分系の接触を疑う

33 ∗ $\displaystyle\int \sqrt{e^x}dx$

n 乗根 → $\dfrac{1}{n}$ 乗

30

[導出]

$$\begin{aligned} a^{\log_b x} &= \left(b^{\log_b a}\right)^{\log_b x} \\ &= b^{\log_b a \log_b x} \\ &= \left(b^{\log_b x}\right)^{\log_b a} \\ &= x^{\log_b a} \end{aligned}$$

$$\begin{aligned} &\int 2^{\log x}dx \\ &= \int x^{\log 2}dx \\ &= \frac{1}{\log 2+1}x^{\log 2+1}+C \\ &= \frac{x\cdot 2^{\log x}}{\log 2+1}+C \quad \blacksquare \end{aligned}$$

31

$$t = \log x$$
$$dt = \frac{1}{x}dx$$

$$\begin{aligned} &\int \frac{\log(\log x)}{x\log x}dx \\ &= \int \frac{\log t}{t}dt \\ &= \int (\log t)\cdot(\log t)'\,dt \\ &= \frac{1}{2}(\log x)^2+C \\ &= \frac{1}{2}\{\log(\log x)\}^2+C \quad \blacksquare \end{aligned}$$

32

$$\begin{aligned} \int \frac{1}{x(\log x)^2}dx &= \int (\log x)'(\log x)^{-2}\,dx \\ &= -(\log x)^{-1}+C \\ &= -\frac{1}{\log x}+C \quad \blacksquare \end{aligned}$$

33

$$\begin{aligned} \int \sqrt{e^x}dx &= \int (e^x)^{\frac{1}{2}}\,dx \\ &= \int e^{\frac{x}{2}}dx \\ &= 2e^{\frac{x}{2}}+C \quad \blacksquare \end{aligned}$$

34 ∗∗∗∗∗　$\displaystyle\int_{-1}^{1}\frac{x^2}{1+e^x}dx$

$f(x)$ が偶関数であるとき
$$\int_{-b}^{b}\frac{f(x)}{1+a^x}dx=\int_{0}^{b}f(x)dx$$

34　$x=-t,\quad dx=-dt$ と置換する．

$$\begin{aligned}\int_{-1}^{1}\frac{x^2}{1+e^x}dx&=\int_{-1}^{0}\frac{x^2}{1+e^x}dx+\int_{0}^{1}\frac{x^2}{1+e^x}dx\\&=\int_{0}^{1}\frac{t^2}{1+e^{-t}}dt+\int_{0}^{1}\frac{x^2}{1+e^x}dx\\&=\int_{0}^{1}\frac{t^2e^t}{1+e^t}dt+\int_{0}^{1}\frac{x^2}{1+e^x}dx\\&=\int_{0}^{1}\frac{x^2\left(1+e^x\right)}{1+e^x}dx\\&=\int_{0}^{1}x^2dx\\&=\frac{1}{3}\quad\blacksquare\end{aligned}$$

35 ∗∗　$\displaystyle\int_{e}^{e^2}x^{\frac{1}{\log x}}dx$

$A^{x\text{ の式}}$
→ 対数をとる

35　$\displaystyle\int_{e}^{e^2}x^{\frac{1}{\log x}}dx$

$$\begin{aligned}y&=x^{\frac{1}{\log x}}\text{ とおくと，}\\\log y&=\log x^{\frac{1}{\log x}}\\&=\frac{1}{\log x}\log x=1\\\therefore y&=e\end{aligned}$$

$$\begin{aligned}\int_{e}^{e^2}x^{\frac{1}{\log x}}dx&=e\int_{e}^{e^2}dx\\&=e\left(e^2-e\right)\\&=e^3-e^2\quad\blacksquare\end{aligned}$$

36 ∗∗∗　$\displaystyle\int\frac{1}{1-e^{-x}}$

微分系の接触を作る

36

$$\begin{aligned}\int\frac{1}{1-e^{-x}}&=\int\frac{e^x}{e^x-1}\\&=\int\frac{\left(e^x-1\right)'}{e^x-1}dx\\&=\log\left|e^x-1\right|+C\quad\blacksquare\end{aligned}$$

37 ∗∗∗∗ $\displaystyle\int x^x(\log x+1)\,dx$

> A^x の式
> → 対数をとる

38 ∗∗∗ $\displaystyle\int_0^{\frac{1}{2}\log 3}\frac{e^x}{1+e^{2x}}dx$

> dx とセットで消える置換を考える

39 ∗ $\displaystyle\int e^{e^x+x}dx$

> 指数はバラす

37 $\displaystyle\int x^x(\log x+1)\,dx$

$y=x^x$とおくと,

$$\begin{aligned}\log y&=\log x^x\\&=x\log x\\\frac{y'}{y}&=\log x+1\\y'&=x^x(\log x+1)\end{aligned}$$

$$\begin{aligned}\int x^x(\log x+1)\,dx&=\int (x^x)'\,dx\\&=x^x+C\end{aligned}$$ ■

38

$t=e^x$

$dt=e^xdx$

x	$0\to\log\sqrt{3}$
t	$1\to\sqrt{3}$

$t=\tan\theta$

$dt=\dfrac{1}{\cos^2\theta}d\theta$

x	$1\to\sqrt{3}$
t	$\frac{\pi}{4}\to\frac{\pi}{3}$

$$\begin{aligned}&\int_0^{\frac{1}{2}\log 3}\frac{e^x}{1+e^{2x}}\\&=\int_0^{\frac{1}{2}\log 3}\frac{1}{1+(e^x)^2}e^xdx\\&=\int_1^{\sqrt{3}}\frac{1}{1+t^2}dt\\&=\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}d\theta\\&=\frac{\pi}{12}\end{aligned}$$ ■

39

$$\begin{aligned}\int e^{e^x+x}dx&=\int e^{e^x}\cdot e^xdx\\&=\int\left(e^{e^x}\right)'dx\\&=e^{e^x}+C\end{aligned}$$ ■

40 ＊＊＊＊　$\displaystyle\int_0^1 \log\left(x^2+1\right)dx$

-2014 旭川医科大学-
単独の log
$\to 1\times\log$ の部分積分

41 ＊＊＊　$\displaystyle\int \frac{\log x}{x^2}dx$

$\log x \to$ 部分積分

42 ＊＊＊　$\displaystyle\int \left(\frac{\log x}{x}\right)^2 dx$

$\log x \to$ ~~部分積分~~
$\to t=\log x$ と置換
$\to$ 指数関数型へ

40

$x=\tan\theta$

$dx=\dfrac{1}{\cos^2\theta}d\theta$

x	$0 \to 1$
θ	$1 \to \frac{\pi}{4}$

$$\begin{aligned}
&\int_0^1 \log\left(x^2+1\right)dx\\
&=\left[x\log\left(x^2+1\right)\right]_0^1-\int_0^1 x\cdot\frac{2x}{x^2+1}dx\\
&=\log 2-2\int_0^1\frac{x^2}{x^2+1}dx\\
&=\log 2-2\left\{\int_0^1 1dx-\int_0^1\frac{1}{x^2+1}dx\right\}\\
&=\log 2-2+2\int_0^{\frac{\pi}{4}}d\theta\\
&=\log 2-2+\frac{\pi}{2}\quad\blacksquare
\end{aligned}$$

41

$$\begin{aligned}
\int\frac{\log x}{x^2}dx&=-\frac{1}{x}\log x+\int\frac{1}{x}\cdot\frac{1}{x}dx\\
&=-\frac{1}{x}\log x-\frac{1}{x}+C\\
&=-\frac{\log x+1}{x}+C\quad\blacksquare
\end{aligned}$$

42　$\displaystyle\int\left(\frac{\log x}{x}\right)^2dx$

$t=\log x,\quad dt=\dfrac{1}{x}dx$ と置換すると，

$$\begin{aligned}
\int\left(\frac{\log x}{x}\right)^2dx&=\int\frac{t^2}{e^t}dt\\
&=\int t^2e^{-t}dt\\
&=-t^2e^{-t}+2\int te^{-t}dt\\
&=-t^2e^{-t}+2\left(-te^{-t}+\int e^{-t}dt\right)\\
&=-t^2e^{-t}-2te^{-t}-2e^{-t}+C\\
&=-\frac{(\log x)^2+2\log x+2}{x}+C\quad\blacksquare
\end{aligned}$$

43 ∗∗∗ $\displaystyle\int \frac{1}{x\left(4-(\log x)^2\right)}dx$

$\log x \to$ ~~部分積分~~
$\to t = \log x$ と置換
($\to$ 指数関数型へ)

44 ∗∗ $\displaystyle\int \frac{(\log x+3)^2}{x}dx$

$\log x$ が入った積分に $\frac{1}{x}$ があるとき
$\to$ 微分系の接触を疑う

45 ∗∗∗∗ $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2}\log\sqrt{1+x^2}dx$

$\log A \to$ 部分積分

43 $t=\log x,\quad dt=\dfrac{1}{x}dx$ と置換すると,

$$\int \frac{1}{x\left(4-(\log x)^2\right)}dx = \int \frac{1}{4-t^2}dt$$
$$= \int \frac{dt}{(2-t)(2+t)}$$
$$= \frac{1}{4}\int\left(\frac{1}{2-t}+\frac{1}{2+t}\right)dt$$
$$= \frac{1}{4}\left(-\log|2-t|+\log|2+t|\right)+C$$
$$= \frac{1}{4}\left(-\log|2-\log x|+\log|2+\log x|\right)+$$

44
$$\int \frac{(\log x+3)^2}{x}dx = \int (\log x+3)'(\log x+3)^2dx$$
$$= \frac{1}{3}(\log x+3)^3+C \qquad ■$$

45
$$\int_1^{\sqrt{3}} \frac{1}{x^2}\log\sqrt{1+x^2}dx = \frac{1}{2}\int_1^{\sqrt{3}}\left(-\frac{1}{x}\right)'\log\left(1+x^2\right)$$
$$= \frac{1}{2}\left[\frac{1}{x}\log\left(1+x^2\right)\right]_1^{\sqrt{3}} - \frac{1}{2}\int_1^{\sqrt{3}}\left(-\frac{1}{x}\right)\frac{2x}{1+x^2}$$
$$= \frac{1}{2}\left(-\frac{1}{\sqrt{3}}\log 4+\log 2\right)+\int_1^{\sqrt{3}}\frac{1}{1+x^2}dx$$

$x=\tan\theta$

$dx = \dfrac{1}{\cos^2\theta}d\theta$

x	1	$\to$	$\sqrt{3}$
θ	$\frac{\pi}{4}$	$\to$	$\frac{\pi}{3}$

$$= \left(\frac{1}{2}-\frac{1}{\sqrt{3}}\right)\log 2+\frac{\pi}{12} \qquad ■$$

46 ∗∗∗　$\displaystyle\int_0^2 \frac{e^x}{e^x+e^{2-x}}dx$

微分系の接触をつくる

47 ∗∗　$\displaystyle\int \frac{3^x}{3^x+\log 3}dx$

微分系の接触をつくる

$(a^x)' = a^x \log a$

46

$$\begin{aligned}\int_0^2 \frac{e^x}{e^x+e^{2-x}}dx &= \int_0^2 \frac{e^{2x}}{e^{2x}+e^2}dx \\ &= \frac{1}{2}\int_0^2 \frac{\left(e^{2x}+e^2\right)'}{e^{2x}+e^2}dx \\ &= \frac{1}{2}\left[\log\left(e^{2x}+e^2\right)\right]_0^2 \\ &= \frac{1}{2}\left\{\log\left(e^4+e^2\right)-\log\left(1+e^2\right)\right\} \\ &= \frac{1}{2}\log\frac{e^2\left(e^2+1\right)}{1+e^2} \\ &= 1 \quad ■\end{aligned}$$

〜*King Property*〜

$$\int_a^b f(x)dx = \int_a^b f(a+b-x)$$

$$2I = \int_a^b f(x)+f(a+b-x)dx$$

King Property を用いた別解を示す

$$\begin{aligned} I &= \int_0^2 \frac{e^x}{e^x+e^{2-x}}dx \\ 2I &= \int_0^2 \frac{e^x}{e^x+e^{2-x}} + \frac{e^{2-x}}{e^{2-x}+e^x} \\ &= \int_0^2 1dx = 2 \\ \therefore I &= 1 \quad ■\end{aligned}$$

47

$$\begin{aligned}\int \frac{3^x}{3^x+\log 3}dx &= \int \frac{1}{\log 3}\frac{(3^x+\log 3)'}{3^x+\log 3}dx \\ &= \frac{1}{\log 3}\log(3^x+\log 3)+C \quad ■\end{aligned}$$

48 ∗∗∗∗∗ $\displaystyle\int_2^4 \frac{\sqrt{\log(9-x)}}{\sqrt{\log(9-x)}+\sqrt{\log(x+3)}}dx$

−1987 *Patnam Competition*−
$\int_a^b f(x)dx$ において
$f(x)+f(a+b-x)$ が単純な形
→ *King Property*

49 ∗∗∗∗∗ $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

$\frac{1}{1+A}$ の形
→ A を $\tan^2\theta$ にする

50 ∗∗∗ $\displaystyle\int \frac{\log x}{(x+1)^3}dx$

$\log x$ → 部分積分

48

$$I=\int_2^4 \frac{\sqrt{\log(9-x)}}{\sqrt{\log(9-x)}+\sqrt{\log(x+3)}}dx$$
$$=\int_2^4 \frac{\sqrt{\log(x+3)}}{\sqrt{\log(x+3)}+\sqrt{\log(9-x)}}dx$$
$$2I=\int_2^4 1dx$$
$$=2$$
$$\therefore I=1$$

49 $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}dx$

$x=\tan\theta$

$dx=\dfrac{1}{\cos^2\theta}d\theta$

x	0	→	1
θ	0	→	$\frac{\pi}{4}$

$\dfrac{\pi}{4}-\theta=t$

$d\theta=-dt$

θ	0	→	$\frac{\pi}{4}$
t	$\frac{\pi}{4}$	→	0

$$=\int_0^{\frac{\pi}{4}} \log(1+\tan\theta)d\theta$$
$$=\int_0^{\frac{\pi}{4}} \log\left(\frac{\cos\theta+\sin\theta}{\cos\theta}\right)d\theta$$
$$=\int_0^{\frac{\pi}{4}} \log\left(\frac{\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)+\sin\theta}{\cos\theta}\right)d$$
$$=\log\sqrt{2}\int_0^{\frac{\pi}{4}} d\theta$$
$$+\int_0^{\frac{\pi}{4}} \log\cos\left(\frac{\pi}{4}-\theta\right)d\theta$$
$$-\int_0^{\frac{\pi}{4}} \log\cos\theta d\theta$$
$$=\frac{\pi}{8}\log 2 \quad ■$$

50 $\displaystyle\int \frac{\log x}{(x+1)^3}dx$

$$=-\frac{1}{2}(x+1)^{-2}\log x+\frac{1}{2}\int\frac{1}{x(x+1)^2}dx$$
$$=-\frac{\log x}{2(x+1)^2}+\frac{1}{2}\left\{\log x-\log(x+1)+\frac{1}{x+1}\right\}+C$$
$$=\frac{1}{2}\left\{\frac{x(x+2)}{(x+1)^2}\log x-\log(x+1)+\frac{1}{x+1}\right\}+C \quad ■$$

51 ∗∗ $\displaystyle\int_e^{e^e} \frac{\log x \cdot \log(\log x)}{x} dx$

−*MIT Integration bee* 2020−
$\int_a^b f(x)dx$ において
$f(x) + f(a+b-x)$ が単純な形
→ *King Property*

51 $\displaystyle\int_e^{e^e} \frac{\log x \cdot \log(\log x)}{x} dx$

$t = \log x$

$dt = \dfrac{1}{x}dx$

x	e	$\to$	e^e
t	1	$\to$	e

$$
\begin{aligned}
&= \int_1^e t \cdot \log t dt \\
&= \left[\frac{1}{2}t^2 \log t dt\right]_1^e - \int_1^e \frac{1}{2}t^2 \cdot \frac{1}{t} dt \\
&= \frac{1}{2}e^2 - \frac{1}{2}\left[\frac{1}{2}t^2\right]_1^e \\
&= \frac{1}{2}e^2 - \frac{1}{4}(e^2 - 1) \\
&= \frac{e^2 - 1}{4} \quad \blacksquare
\end{aligned}
$$

52 ∗ $\displaystyle\int_0^2 x \log(x+1) dx$

-2017 弘前大学-
$\log A \to$ 部分積分

52 $\displaystyle\int_0^2 x \log(x+1) dx$

$$
\begin{aligned}
&= \left[\frac{x^2}{2}\log(x+1) - \int \frac{x^2}{2} \cdot \frac{1}{x+1} dx\right]_0^2 \\
&= \left[\frac{x^2}{2}\log(x+1) - \int \frac{1}{2}\left(\frac{x^2 - 1 + 1}{x+1}\right) dx\right]_0^2 \\
&= \left[\frac{x^2}{2}\log(x+1) - \int \frac{1}{2}\left(\frac{(x+1)(x-1) + 1}{x+1}\right) dx\right]_0^2 \\
&= \left[\frac{x^2}{2}\log(x+1) - \int \frac{1}{2}\left(x - 1 + \frac{1}{x+1}\right) dx\right]_0^2 \\
&= \left[\frac{x^2}{2}\log(x+1) - \frac{(x-1)^2}{4} - \frac{\log|x+1|}{2}\right]_0^2 \\
&= \left(2\log 3 - \frac{1}{4} - \frac{\log 3}{2}\right) - \left(0 - \frac{1}{4} - 0\right) \\
&= \frac{3\log 3}{2} \quad \blacksquare
\end{aligned}
$$

53 ∗∗ $\displaystyle\int_0^1 x\log\left(x^2+1\right)dx$

log A の形をつくる

54 ∗ $\displaystyle\int_1^e \frac{\log x}{x^3}dx$

$\log x \to$ 部分積分

55 ∗∗ $\displaystyle\int \frac{1}{e^x(e^x+1)}dx$

部分分数分解を繰り返す

53

$u = x^2+1$
$du = 2xdx$

x	$0 \to 1$
u	$1 \to 2$

$$\begin{aligned}
&\int_0^1 x\log\left(x^2+1\right)dx \\
&= \int_1^2 \frac{1}{2}\log u du \\
&= \frac{1}{2}\left[u(\log u - 1)\right]_1^2 \\
&= \frac{1}{2}(2\log 2 - 2 + 1) \\
&= \frac{1}{2}(2\log 2 - 1) \quad \blacksquare
\end{aligned}$$

54

$$\begin{aligned}
\int_1^e \frac{\log x}{x^3}dx &= \int_1^e \log x \cdot x^{-3}dx \\
&= \int_1^e \log x \cdot \left(\frac{x^{-2}}{-2}\right)' dx \\
&= \left[\log x \cdot \left(\frac{x^{-2}}{-2}\right) - \int \frac{1}{x}\cdot\left(\frac{x^{-2}}{-2}\right)dx\right]_1^e \\
&= \left[-\frac{1}{2x^2}\log x - \int \frac{x^{-3}}{-2}dx\right]_1^e \\
&= \left[-\frac{1}{2x^2}\log x - \frac{1}{4x^2}\right]_1^e \\
&= -\frac{1}{2e^2} - \frac{1}{4e^2} + \frac{1}{4} \\
&= -\frac{3}{4e^2} + \frac{1}{4} \quad \blacksquare
\end{aligned}$$

55

$$\begin{aligned}
\int \frac{1}{e^x(e^x+1)}dx &= \int \left(\frac{1}{e^x} - \frac{1}{e^x+1}\right)dx \\
&= -e^{-x} - \int \frac{1}{e^x+1}dx
\end{aligned}$$

$t = e^x$
$dt = e^x dx$

$$\begin{aligned}
&= -e^{-x} - \int \frac{1}{t+1}\cdot\frac{dt}{t} \\
&= -e^{-x} - \int \left(\frac{1}{t} - \frac{1}{t+1}\right)dt \\
&= -e^{-x} - \log|t| + \log|t+1| + C \\
&= -e^{-x} - \log e^x + \log(e^x+1) + C \\
&= -e^{-x} - x + \log(e^x+1) + C \quad \blacksquare
\end{aligned}$$

0.3　三角関数の積分

56 ∗∗∗　$\displaystyle\int \frac{1}{\sin x}dx$

$f(\cos x)\sin x$
$\to \cos x = t$ と置換

56　$\displaystyle\int \frac{1}{\sin x}dx$

$$\begin{aligned}\frac{1}{\sin x} &= \frac{\sin x}{\sin^2 x}\\ &= \frac{\sin x}{1-\cos^2 x}\\ &= \frac{1}{t^2-1}\\ &= \frac{1}{(t+1)(t-1)}\\ &= \frac{1}{2}\left(\frac{1}{t-1}-\frac{1}{t+1}\right)\end{aligned}$$

$$\begin{aligned}t &= \cos x\\ dt &= -\sin x dx\end{aligned}$$
$$\left|\frac{\cos x - 1}{\cos x + 1}\right| = \frac{1-\cos x}{\cos x + 1}$$

$$\begin{aligned}\int \frac{1}{\sin x}dx &= \frac{1}{2}\int\left(\frac{1}{t-1}-\frac{1}{t+1}\right)dt\\ &= \frac{1}{2}\log\left|\frac{t-1}{t+1}\right| + C\\ &= \frac{1}{2}\log\frac{1-\cos x}{1+\cos x} + C \quad \blacksquare\end{aligned}$$

57 ∗∗　$\displaystyle\int \tan^3 x dx$

三角関数は 2 乗の形を作る

57

$$\begin{aligned}\int \tan^3 x dx &= \int \tan x \cdot \tan^2 x dx\\ &= \int \tan x \cdot (\frac{1}{\cos^2 x} - 1)dx\\ &= \int \tan x \cdot \frac{1}{\cos^2 x} + \frac{-\sin x}{\cos x}dx\\ &= \int \tan x \cdot (\tan x)' + \frac{(\cos x)'}{\cos x}dx\\ &= \frac{1}{2}\tan^2 x + \log|\cos x| + C \quad \blacksquare\end{aligned}$$

58 ＊＊＊＊ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x + 1} dx$

-2023 富山大学-
ユニバーサル三角関数置換積分

58 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x + 1} dx$

$t = \tan \dfrac{x}{2}$ と置換する

$$\begin{aligned}
\sin x &= \frac{\sin x}{1} \\
&= \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} \\
&= \frac{2\tan\frac{x}{2}}{1 + \tan^2\frac{x}{2}} \\
&= \frac{2t}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
\cos x &= \frac{\cos x}{1} \\
&= \frac{\cos^2\frac{x}{2} - \sin^2\frac{x}{2}}{\cos^2\frac{x}{2} + \sin^2\frac{x}{2}} \\
&= \frac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}} \\
&= \frac{1-t^2}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} \\
&= \frac{2t}{1+t^2} \cdot \frac{1+t^2}{1-t^2} \\
&= \frac{2t}{1-t^2}
\end{aligned}$$

$t = \tan^2 \dfrac{x}{2}$ を微分して,

$$\begin{aligned}
\frac{dt}{dx} &= \frac{1}{\cos^2\frac{x}{2}} \cdot \frac{1}{2} \\
&= \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) \\
&= \frac{1+t^2}{2}
\end{aligned}$$

$$dx = \frac{2}{1+t^2} dt$$

x	0	$\to$	$\frac{\pi}{2}$
t	0	$\to$	1

$$\sin x = \frac{2t}{1+t^2}$$
$$\cos x = \frac{1-t^2}{1+t^2}$$
$$\tan x = \frac{2t}{1-t^2}$$

$$\begin{aligned}
&\int_0^{\frac{\pi}{2}} \frac{1}{\sin x + \cos x + 1} dx \\
&= \int_0^1 \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1} \cdot \frac{2}{1+t^2} dt \\
&= \int_0^1 \frac{2}{2t + (1-t^2) + (1+t^2)} dt \\
&= \int_0^1 \frac{1}{1+t} dt \\
&= \left[\log(1+t)\right]_0^1 = \log 2 \quad \blacksquare
\end{aligned}$$

59 ＊＊ $\displaystyle\int \sin(\log x)\,dx$

三角関数の中身が複雑
→ 丸ごと置換

60 ＊＊ $\displaystyle\int \cos 2x \cos 4x dx$

三角関数の積
→ 積和の公式

61 ＊＊＊＊＊ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{(\cos x)^{\sqrt{3}}}{(\sin x)^{\sqrt{3}} + (\cos x)^{\sqrt{3}}} dx$

$\sin x$ と $\cos x$ の対称性に注目

$$\sin\left(\frac{\pi}{2} - x\right) = \cos x$$

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x$$

59

$$t = \log x$$
$$x = e^t$$
$$dx = e^t dt$$

$$\begin{aligned} &\int \sin(\log x)\,dx \\ &= \int \sin t e^t dt \\ &= \frac{1}{2} e^t (\sin t - \cos t) + C \\ &= \frac{1}{2} x\{\sin(\log x) - \cos(\log x)\} + C \quad \blacksquare \end{aligned}$$

60

$$\begin{aligned} \int \cos 2x \cos 4x dx &= \frac{1}{2} \int (\cos 6x + \cos 2x)\,dx \\ &= \frac{1}{2}\left(\frac{1}{6}\sin 6x + \frac{1}{2}\sin 2x\right) + C \\ &= \frac{1}{12}\sin 6x + \frac{1}{4}\sin 2x + C \quad \blacksquare \end{aligned}$$

61

$$x = \frac{\pi}{2} - t$$
$$dx = -dt$$

x	0	$\to$	$\frac{\pi}{2}$
t	$\frac{\pi}{2}$	$\to$	0

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \frac{(\cos x)^{\sqrt{3}}}{(\sin x)^{\sqrt{3}} + (\cos x)^{\sqrt{3}}} dx \\ &= \int_0^{\frac{\pi}{2}} \frac{(\sin t)^{\sqrt{3}}}{(\cos t)^{\sqrt{3}} + (\sin t)^{\sqrt{3}}} dt \\ &= \int_0^{\frac{\pi}{2}} \frac{(\cos x)^{\sqrt{3}}}{(\sin x)^{\sqrt{3}} + (\cos x)^{\sqrt{3}}} dx \\ \therefore 2I &= \int_0^{\frac{\pi}{2}} 1 dx = \frac{\pi}{2} \\ I &= \frac{\pi}{4} \quad \blacksquare \end{aligned}$$

King Property による解法もある

62 ∗∗ $\displaystyle\int \cos x \cdot \cos 2x \cdot \cos 3x dx$

-東京理科大学-
三角関数の積
→ 積和の公式

62
$$\cos x\cos 2x\cos 3x = \frac{\cos 3x+\cos x}{2}\cdot\cos 3x$$
$$= \frac{1}{2}(\cos^2 3x+\cos x\cos 3x)$$
$$= \frac{1}{2}\left(\frac{1+\cos 6x}{2}+\frac{\cos 4x+\cos 2x}{2}\right)$$
$$= \frac{1}{4}(1+\cos 2x+\cos 4x+\cos 6x)$$

$$\therefore \int \cos x\cos 2x\cos 3x dx$$
$$= \frac{1}{4}(x+\frac{1}{2}\sin 2x+\frac{1}{4}\sin 4x+\frac{1}{6}\sin 6x)+C$$
$$= \frac{1}{48}(12x+6\sin 2x+3\sin 4x+2\sin 6x)+C$$ ■

63 ∗∗∗∗ $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos x\sin x}{\cos^4 x+\sin^4 x}dx$

対称性を崩し
$\cos^n x$ で割って $\tan x$ の話題へ

63

$\tan^2 x = u$

$(\tan^2 x)' dx = du$

x	0	$\to$	$\frac{\pi}{4}$
u	0	$\to$	1

$u = \tan\theta$

$du = \dfrac{d\theta}{\cos^2\theta}$

u	0	$\to$	1
θ	0	$\to$	$\frac{\pi}{4}$

$$\int_0^{\frac{\pi}{4}} \frac{\cos x\sin x}{\cos^4 x+\sin^4 x}dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{\cos^4 x}}{\frac{1}{\cos^4 x}}\frac{\cos x\sin x}{\cos^4 x+\sin^4 x}dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{\tan x\cdot\frac{1}{\cos^2 x}}{1+\tan^4 x}dx$$
$$= \int_0^{\frac{\pi}{4}} \frac{\frac{1}{2}(\tan^2 x)'}{1+(\tan^2 x)^2}dx$$
$$= \int_0^{1} \frac{1}{2}\frac{1}{1+u^2}du$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}\frac{1}{1+\tan^2\theta}\frac{d\theta}{\cos^2\theta}$$
$$= \frac{1}{2}\int_0^{\frac{\pi}{4}} d\theta = \frac{\pi}{8}$$ ■

64 ＊＊＊＊＊ $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x^2}{(x\sin x+\cos x)^2}dx$

微分形の接触を作る

$(x\sin x+\cos x)'$

$=\sin x+x\cos x-\sin x$

$=x\cos x$

64 $\displaystyle\int_0^{\frac{\pi}{4}} \frac{x^2}{(x\sin x+\cos x)^2}dx$

$$=\int_0^{\frac{\pi}{4}} \underbrace{\left(\frac{x}{\cos x}\right)}_{\text{微分}}\underbrace{\frac{x\cos x}{(x\sin x+\cos x)}}_{\text{積分}}dx$$

$$=\left[-\frac{x}{\cos x}\frac{1}{x\sin x+\cos x}\right]_0^{\frac{\pi}{4}}+\int_0^{\frac{\pi}{4}}\frac{\cancel{\cos x+x\sin x}}{\cos^2 x}\frac{dx}{\cancel{x\sin x+\cos x}}$$

$$=-\frac{\frac{\pi}{4}}{\frac{1}{\sqrt{2}}}\frac{1}{\frac{\pi}{4}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}+[\tan x]_0^{\frac{\pi}{4}}$$

$$=-\frac{\pi}{4}\frac{1}{\frac{\pi}{4}\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}}+1$$

$$=-\frac{\pi}{\pi+4}+1=\frac{4-\pi}{4+\pi} \quad ■$$

65 ＊＊＊＊＊ $\displaystyle\int_{-1}^{1} \frac{\sin^2(\pi x)}{1+e^x}dx$

$\displaystyle\int_a^b f(x)dx$ において

$f(x)+f(a+b-x)$ が単純な形

$\rightarrow$ *King Property*

65

$$I=\int_{-1}^{1}\frac{\sin^2(\pi x)}{1+e^x}dx$$

$$2I=\int_{-1}^{1}\left\{\frac{\sin^2(\pi x)}{1+e^x}\frac{\sin^2(\pi x)}{1+e^{-x}}\right\}dx$$

$$=\int_{-1}^{1}\frac{\cancel{(1+e^x)}\sin^2(\pi x)}{\cancel{1+e^x}}dx$$

$$=\int_{-1}^{1}\sin^2(\pi x)dx$$

$$=\int_{-1}^{1}\frac{1-\cos(2\pi x)}{2}dx$$

$$=\left[\frac{1}{2}x-\frac{1}{4\pi}\sin(2\pi x)\right]_{-1}^{1}=1$$

$$\therefore I=\frac{1}{2} \quad ■$$

66 ＊＊ $\displaystyle\int_0^{\pi} e^{2x}\sin x dx$

(指数関数)×(三角関数)

→ 部分積分して同形出現

66

$$I=\int_0^{\pi} e^{2x}\sin x dx$$

$$=\left[\frac{1}{2}e^{2x}\sin x\right]_0^{\pi}-\frac{1}{2}\int_0^{\pi}e^{2x}\cos x dx$$

$$=-\frac{1}{2}\left\{\left[\frac{1}{2}e^{2x}\cos x\right]_0^{\pi}+\frac{1}{2}\underline{\int_0^{\pi}e^{2x}\sin x dx}\right\}$$

$$=-\frac{1}{2}\left(-\frac{1}{2}e^{2\pi}-\frac{1}{2}+\frac{1}{2}I\right)$$

$$I=\frac{1}{5}\left(e^{2\pi}+1\right) \quad ■$$

67 ＊＊＊＊＊ $\displaystyle\int_0^{\pi} \frac{x\sin x}{3+\cos 2x}dx$

$$\int_0^{\pi} xf(\sin x)dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

68 ＊＊＊ $\displaystyle\int (x^2-2x)\cos 2xdx$

$\underset{微}{\underline{(整式)}} \times \underset{積}{\underline{(三角関数)}}$

→ 部分積分を繰り返す

67

$x=\pi-t$

$dx=-dt$

x	0	$\to$	π
t	π	$\to$	0

$t=\cos x$

$dt=-\sin xdx$

x	0	$\to$	π
t	1	$\to$	-1

$t=\tan\theta$

$dt=\dfrac{1}{\cos^2\theta}d\theta$

t	-1	$\to$	1
θ	$-\frac{\pi}{4}$	$\to$	$\frac{\pi}{4}$

−証明−

$$\begin{aligned}I &= \int_0^{\pi} xf(\sin x)dx\\ &= \int_{\pi}^{0} (\pi-t)f(\sin(\pi-t))(-dt)\\ &= \int_0^{\pi} (\pi-t)f(\sin t)dt\\ &= \pi\int_0^{\pi} f(\sin t)dt\\ &\qquad -\int_0^{\pi} tf(\sin t)dt\end{aligned}$$

これを用いて

$$\begin{aligned}&\int_0^{\pi} \frac{x\sin x}{3+\cos 2x}dx\\ &= \frac{\pi}{2}\int_0^{\pi}\frac{\sin x}{3+\cos 2x}dx\\ &= \frac{\pi}{4}\int_0^{\pi}\frac{\sin x}{1+\cos^2 x}dx\\ &= \frac{\pi}{4}\int_0^{\pi}\frac{1}{1+t^2}dt\\ &= \frac{\pi}{4}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\theta = \frac{\pi}{8} \qquad \blacksquare\end{aligned}$$

68

$$\begin{aligned}&\int (x^2-2x)\cos 2xdx\\ &= (x^2-2x)\frac{1}{2}\sin 2x - \int (2x-2)\cdot\frac{1}{2}\sin 2xdx\\ &= (x^2-2x)\frac{1}{2}\sin 2x - (2x-2)(-\frac{1}{4}\cos 2x)\\ &\qquad + \int 2(-\frac{1}{4}\cos 2x)dx\\ &= (x^2-2x)\frac{1}{2}\sin 2x - (2x-2)(-\frac{1}{4}\cos 2x)\\ &\qquad - \frac{1}{2}\cdot\frac{1}{2}\sin 2x + C\\ &= \frac{1}{4}\left\{(2x^2-4x-1)\sin 2x + 2(x-1)\cos 2x\right\} + C \ \blacksquare\end{aligned}$$

69 ∗∗∗　$\displaystyle\int_0^{\frac{\pi}{2}} \sin^8 x dx$

~~倍角の公式で次数下げ~~
→ 漸化式の立式

69

$$I_8 = \int_0^{\frac{\pi}{2}} \sin^8 x dx$$

$I_n = \displaystyle\int_0^{\frac{\pi}{2}} \sin^n x dx$ とおくと,

$$= \left[-\cos x \sin^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}} \cos^2 x \sin^{n-2} x dx$$

$$= (n-1)\int_0^{\frac{\pi}{2}} (1-\sin^2 x)\sin^{n-2} x dx$$

$$= (n-1)\{I_{n-2} - I_n\}$$

$$\therefore I_n = \frac{n-1}{n} I_{n-2}$$

$I_8 = \dfrac{7}{8} I_6 = \dfrac{7}{8}\cdot\dfrac{5}{6} I_4 = \cdots = \dfrac{7}{8}\cdot\dfrac{5}{6}\cdot\dfrac{3}{4}\cdot\dfrac{1}{2} I_0$ となるので,

$I_0 = \displaystyle\int_0^{\frac{\pi}{2}} dx = \dfrac{\pi}{2}$ より,

$$I_8 = \frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{35}{256}\pi \quad ■$$

70 ∗∗∗∗　$\displaystyle\int_0^{\frac{\pi}{4}} \tan^8 x dx$

$\tan x$ のたんは異端児の”たん”
→ $\tan^2 x$ を変形すると $(\tan x)'$ が出現

70

$$J_8 = \int_0^{\frac{\pi}{4}} \tan^8 x dx$$

$$J_n = \int_0^{\frac{\pi}{4}} \left(\frac{1}{\cos^2 x} - 1\right)\tan^{n-2} x dx$$

$$= \int_0^{\frac{\pi}{4}} (\tan x)' \tan^{n-2} x dx - J_{n-2}$$

$$= \left[\frac{\tan^{n-1} x}{n-1}\right]_0^{\frac{\pi}{4}} - J_{n-2}$$

$$\therefore J_n = -J_{n-2} + \frac{1}{n-1}$$

$$J_8 = -J_6 + \frac{1}{7} = -(-J_4 + \frac{1}{5}) + \frac{1}{7}$$

$$= J_4 - \frac{1}{5} + \frac{1}{7} = -J_2 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7}$$

$$= -(-J_0 + \frac{1}{1}) + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} = J_0 - 1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7}$$

$J_0 = \displaystyle\int_0^{\frac{\pi}{4}} dx = \dfrac{\pi}{4}$ と計算して

$$J_8 = \frac{\pi}{4} - \frac{76}{105} \quad ■$$

71 ∗∗∗ $\displaystyle\int_0^{\frac{\pi}{3}} \frac{1}{\cos x}dx$

-2019 京都大学-
$f(\sin x)\cos x$ の形
$\to \sin x = t$ と置換

71 $\displaystyle\int \frac{1}{\cos x}dx = \int \frac{\cos x}{\cos^2 x}dx$
$\displaystyle= \int \frac{\cos x}{1-\sin^2 x}dx$

$t = \sin x$
$dt = \cos x dx$

$$
\begin{aligned}
&= \int \frac{1}{1-t^2}dt \\
&= \int \frac{1}{(1+t)(1-t)}dt \\
&= \frac{1}{2}\int \left(\frac{1}{1+t}+\frac{1}{1-t}\right)dt \\
&= \frac{1}{2}\{\log|1+t| - \log|1-t|\} + C \\
&= \frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C \\
&= \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right) + C \quad \blacksquare
\end{aligned}
$$

72 $\displaystyle\int \frac{1}{\cos^3 x}dx$

-横浜市立大学-
$\displaystyle\int \frac{1}{\cos x}dx$ にならい, 分母分子に $\cos x$ を掛ける
$\to$ 微分系の接触

72 $\sin x = t, \cos x dx = dt$ と置換する

$$
\begin{aligned}
\int \frac{1}{\cos^3 x}dx &= \int \frac{\cos x}{\cos^4 x}dx \\
&= \int \frac{\cos x}{(1-\sin^2 x)^2}dx \\
&= \int \frac{1}{(1-t^2)^2}dt \\
&= \int \frac{1}{(1+t)^2(1-t)^2}dt \\
&= \frac{1}{4}\int \left\{\frac{1}{1+t}+\frac{1}{1-t}+\frac{1}{(1+t)^2}+\frac{1}{(1-t)^2}\right\} \\
&= \frac{1}{4}\left\{\log\left|\frac{1+t}{1-t}\right| + \frac{2t}{(1+t)(1-t)}\right\} + C \\
&= \frac{1}{4}\left\{\log\left|\frac{1+t}{1-t}\right| + \frac{2t}{(1+t)(1-t)}\right\} + C \\
&= \frac{1}{4}\left(\log\frac{1+\sin x}{1-\sin x} + \frac{2\sin x}{\cos^2 x}\right) + C \quad \blacksquare
\end{aligned}
$$

73 ∗∗∗ $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + 2\cos x}dx$

> 対称性を用いずとも
> I を計算出来る一次結合で表す

73

$$J = \int_0^{\frac{\pi}{2}} \frac{(\sin x + 2\cos x)'}{\sin x + 2\cos x}dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - 2\sin x}{\sin x + 2\cos x}dx$$
$$= [\log|\sin x + 2\cos x]_0^{\frac{\pi}{2}}$$
$$= -\log 2$$
$$5I = 5 \times \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + 2\cos x}dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x - 2\sin x}{\sin x + 2\cos x} + 2dx$$
$$= [-\log 2 + 2x]_0^{\frac{\pi}{2}}$$
$$= \pi - \log 2$$
$$\therefore I = \frac{\pi}{5} - \frac{1}{5}\log 2 \quad ■$$

三角関数の合成を用いた別解を示す.

> $\cos\alpha = \dfrac{1}{\sqrt{5}}$
> $\sin\alpha = \dfrac{2}{\sqrt{5}}$

> $x = t - \alpha$
> $dx = dt$
>
> | x | $0 \to$ | $\frac{\pi}{2}$ |
> |---|---|---|
> | t | $\alpha \to$ | $\frac{\pi}{2} + \alpha$ |

$$\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{5}(\frac{1}{\sqrt{5}}\sin x + \frac{2}{\sqrt{5}}\cos x)}dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{5}(\cos\alpha\sin x + \sin\alpha\cos x)}dx$$
$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{5}(\sin(x+\alpha))}dx$$
$$= \int_\alpha^{\frac{\pi}{2}+\alpha} \frac{\cos(t-\alpha)}{\sqrt{5}\sin t}dt$$
$$= \int_\alpha^{\frac{\pi}{2}+\alpha} \frac{\cos t\cos\alpha + \sin t\sin\alpha}{\sqrt{5}\sin t}dt$$
$$= \int_\alpha^{\frac{\pi}{2}+\alpha} \frac{\frac{1}{\sqrt{5}}\cos t + \frac{2}{\sqrt{5}}\sin t}{\sqrt{5}\sin t}dt$$
$$= \frac{1}{5}\int_\alpha^{\frac{\pi}{2}+\alpha} \frac{\cos t + 2\sin t}{\sin t}dt$$
$$= \frac{1}{5}\left\{\int_\alpha^{\frac{\pi}{2}+\alpha} \frac{\cos t}{\sin t}dt + 2\int_\alpha^{\frac{\pi}{2}+\alpha} dt\right\}$$
$$= \frac{1}{5}\left\{\log\sin\left(\alpha + \frac{\pi}{2}\right) - \log\sin\alpha + 2(\frac{\pi}{2})\right\}$$
$$= \frac{1}{5}\left\{\log\left(\frac{1}{\sqrt{5}}\right) - \log\left(\frac{2}{\sqrt{5}}\right) + \pi\right\}$$
$$= \frac{\pi}{5} - \frac{1}{5}\log 2 \quad ■$$

74 ＊＊＊＊ $\displaystyle\int \frac{\tan^{2024} x}{\sin^2 x}dx$

−*MIT Integration bee* 2024−
微分系の接触を作る

75 ＊＊＊ $\displaystyle\int_0^{2\pi} (\sin x + \cos x)^{11} dx$

−*MIT Integration bee* 2024−
積分区間を分ける

76 ＊＊＊＊＊ $\displaystyle\int \cos^x x \{\log(\cos x) - x\tan x\}\, dx$

−*MIT Integration bee* 2024−
対数微分を用いて
微分系の接触を作る

74

$$\begin{aligned}\int \frac{\tan^{2024} x}{\sin^2 x}dx &= \int \frac{1}{\sin^2 x}\frac{\sin^{2024} x}{\cos^{2024} x}dx \\ &= \int \frac{\sin^{2022} x}{\cos^{2024} x}dx \\ &= \int \frac{\tan^{2022} x}{\cos^2 x}dx \\ &= \frac{1}{2023}\tan^{2023} x + C \quad ■\end{aligned}$$

75 $\displaystyle\int_0^{2\pi} (\sin x + \cos x)^{11} dx$

$$\begin{aligned}&= \int_0^{\pi} (\sin x + \cos x)^{11} dx + \int_{\pi}^{2\pi} (\sin x + \cos x)^{11} dx \\ &= \int_0^{\pi} (\sin x + \cos x)^{11} dx + \int_0^{\pi} \{\sin(x+\pi) + \cos(x+\pi)\}^{11}\, dx \\ &= \int_0^{\pi} (\sin x + \cos x)^{11} dx - \int_0^{\pi} (\sin x + \cos x)^{11}\, dx \\ &= 0 \quad ■\end{aligned}$$

76 $\displaystyle\int \cos^x x \{\log(\cos x) - x\tan x\}\, dx$

$$\begin{aligned} y &= \cos^x x \\ \log y &= x\log(\cos x) \\ \frac{y'}{y} &= \log(\cos x) + x\frac{1}{\cos x}(-\sin x) \\ &= \log(\cos x) - x\tan x \end{aligned}$$

$$\begin{aligned}\int y\cdot\frac{y'}{y}dx &= y + C \\ &= \cos^x x + C \quad ■\end{aligned}$$

77 ∗∗∗ $\displaystyle\int \tan^5 x dx$

三角関数は 2 乗に強い

77 $\displaystyle\int \tan^5 x dx$

$I_n = \displaystyle\int \tan^n x dx$ とおく

$$
\begin{aligned}
I_n &= \int \tan^n x dx \quad (n \geq 2) \\
&= \int \tan^2 x \tan^{n-2} dx \\
&= \int \left(\frac{1}{\cos^2 x} - 1 \right) \cdot \tan^{n-2} dx \\
&= \int (\tan x)' \tan^{n-2} x dx - \int \tan^{n-2} x dx \\
&= \frac{1}{n-1} \tan^{n-1} - I_{n-2} \\
\therefore I_5 &= \frac{1}{4} \tan^4 x - I_3 \\
&= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + I_1 \\
&= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x - \log|\cos x| + C \qquad \blacksquare
\end{aligned}
$$

78 ∗∗∗ $\displaystyle\int e^x \sin^2 x dx$

(指数関数)×(三角関数)
→ 部分積分して同形出現

78

$$
\begin{aligned}
\int e^x \sin^2 x dx &= e^x \sin^2 x - \int e^x \cdot 2 \sin x \cos x dx \\
&= e^x \sin^2 x - (e^x \sin 2x - 2 \int e^x \cos 2x dx) \\
&= e^x (\sin^2 x - \sin 2x + 2) - 4 \underbrace{\int e^x \sin^2 x dx}_{I}
\end{aligned}
$$

$$
I = \frac{1}{5} e^x (\sin^2 x - \sin 2x + 2) + C \qquad \blacksquare
$$

79 ∗∗∗ $\displaystyle\int_0^{\frac{\pi}{6}} \cos^6 x dx$

-同志社大学-
三角関数は 2 乗に強い

80 ∗∗∗ $\displaystyle\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx$

$f(\cos x)\sin x$
$\rightarrow t = \cos x$ と置換

81 ∗ $\displaystyle\int_0^{\pi} \frac{\sin x}{1-\cos^2 x} dx$

-甲南大学-
三角関数は 2 乗に強い

79

$$\begin{aligned}
&\int_0^{\frac{\pi}{6}} \cos^6 x dx \\
&= \int_0^{\frac{\pi}{6}} \left(\frac{1+\cos 2x}{2}\right)^3 dx \\
&= \frac{1}{8}\int_0^{\frac{\pi}{6}} 1+3\cos 2x+3\cos^2 2x+\cos^3 2x dx \\
&= \frac{1}{8}\int_0^{\frac{\pi}{6}} 1+3\cos 2x+3\cdot\frac{1+\cos 4x}{2}+\frac{\cos 6x+3\cos 2x}{4} dx \\
&= \frac{1}{8}\left[\frac{5}{2}x+\frac{15}{8}\cdot\frac{\sqrt{3}}{2}+\frac{3}{8}\cdot\frac{\sqrt{3}}{2}+\frac{1}{24}\cdot 0\right] \\
&= \frac{5}{96}\pi+\frac{9\sqrt{3}}{64} \quad ■
\end{aligned}$$

80

$t = \cos x$
$dt = -\sin x dx$

x	0	$\rightarrow$	π
t	1	$\rightarrow$	-1

$t = \tan\theta$
$dt = \dfrac{d\theta}{\cos^2\theta}$

t	0	$\rightarrow$	1
θ	0	$\rightarrow$	$\frac{\pi}{4}$

$$\begin{aligned}
&\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} dx \\
&= \int_{-1}^{1} \frac{1}{1+t^2} dt \\
&= 2\int_0^{1} \frac{1}{1+t^2} dt \\
&= 2\int_0^{\frac{\pi}{4}} d\theta \\
&= \frac{\pi}{2} \quad ■
\end{aligned}$$

81

$$\begin{aligned}
\int \frac{\sin x}{1-\cos^2 x} dx &= \frac{1}{2}\int \frac{\sin x}{1+\cos x}+\frac{\sin x}{1-\cos x} dx \\
&= \frac{1}{2}\log\left|\frac{1-\cos x}{1+\cos x}\right| \quad ■
\end{aligned}$$

82 ∗∗∗ $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\cos 3x}{\cos 2x}\right)^2 dx$

-兵庫県立大学-
三角関数は 2 乗に強い

83 ∗∗∗∗∗ $\displaystyle\int_0^{\pi} \frac{1}{1+(\sin x)^{\cos x}}dx$

被積分関数に三角関数が多いとき
→ *King Property*

84 ∗∗∗∗∗ $\displaystyle\int_0^{\frac{\pi}{6}} \frac{\sin^3 3x}{\sin^3 3x+\cos^3 3x}dx$

-藤田医科大学-
$\sin x$ と $\cos x$ の対称性に着目

82

$$\begin{aligned}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\cos 3x}{\cos 2x}\right)^2 dx &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\frac{\sin 2x\cos x+\cos 2x\sin x}{\sin 2x}\right)^2 dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(\cos x+\frac{2\cos^2 x-1}{2\cos x}\right)^2 dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \left(2\cos x-\frac{1}{2\cos x}\right)^2 dx \\ &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} 2\cos 2x+\frac{1}{4\cos^2 x}dx \\ &= \left[\sin 2x+\frac{1}{4}\tan x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \frac{\sqrt{3}}{6} \quad \blacksquare\end{aligned}$$

83

$I=\displaystyle\int_0^{\pi} \frac{1}{1+(\sin x)^{\cos x}}dx$ とおく

$$\begin{aligned}2I &= \int_0^{\pi}\left\{\frac{1}{1+(\sin x)^{\cos x}}+\frac{1}{1+(\sin x)^{-\cos x}}\right\} \\ &= \int_0^{\pi} dx \\ &= \pi \\ I &= \frac{\pi}{2} \quad \blacksquare\end{aligned}$$

84

$\dfrac{\pi}{6}-x=t$

$dx=-dt$

x	0	$\to$	$\frac{\pi}{6}$
t	$\frac{\pi}{6}$	$\to$	0

$$\begin{aligned}I &= \int_0^{\frac{\pi}{6}} \frac{\sin^3 3x}{\sin^3 3x+\cos^3 3x}dx \\ &= \int_0^{\frac{\pi}{6}} \frac{\cos^3 3t}{\cos^3 3t+\sin^3 3t}dt \\ 2I &= \int_0^{\frac{\pi}{6}} \frac{\sin^3 3x+\cos^3 3x}{\sin^3 3x+\cos^3 3x}dx \\ &= \int_0^{\frac{\pi}{6}} dx \\ &= \frac{\pi}{6} \quad \blacksquare\end{aligned}$$

85 ∗∗∗ $\displaystyle\int_0^{2\pi} \sin(\sin x - x)dx$

−*MIT Integration bee* 2020−
被積分関数に三角関数が多いとき
→ *King Property*

86 ∗∗ $\displaystyle\int x\sin x\cos x dx$

三角関数の積
→ 和の形にする

87 ∗∗∗∗∗ $\displaystyle\int_0^{2\pi} \frac{\cos x}{\sin x + \cos x}dx$

$\sin x$ と $\cos x$ の対称性に着目

85

$$\begin{aligned}2I &= \int_0^{2\pi} \sin(\sin x - x) + \sin(\sin(2\pi - x) - (2\pi - x))dx\\ &= \int_0^{2\pi} \sin(\sin x - x) - \sin(\sin x - x)dx\\ &= 0\\ I &= 0 \quad ■\end{aligned}$$

86

$$\begin{aligned}\int x\sin x\cos x dx &= \frac{1}{2}\int x\sin 2x dx\\ &= \frac{1}{2}\left(-\frac{1}{2}x\cos 2x + \frac{1}{2}\int \cos 2x dx\right)\\ &= -\frac{1}{4}x\cos 2x + \frac{1}{4}\frac{1}{2}\sin 2x + C\\ &= -\frac{1}{4}x\cos 2x + \frac{1}{8}\sin 2x + C \quad ■\end{aligned}$$

87

$$x = -\frac{\pi}{2} - t$$
$$dx = -dt$$

x	0	$\to$	$\frac{\pi}{2}$
t	$\frac{\pi}{2}$	$\to$	0

$$\begin{aligned}I &= \int_0^{2\pi} \frac{\cos x}{\sin x + \cos x}dx\\ &= \int_0^{2\pi} \frac{\sin t}{\sin t + \cos t}dt\\ 2I &= \int_0^{2\pi} \frac{\cos x + \sin x}{\sin x + \cos x}dx\\ &= \frac{\pi}{2}\\ I &= \frac{\pi}{4} \quad ■\end{aligned}$$

(別解)

$$\begin{aligned}I &= \frac{1}{2}\int_0^{2\pi} \frac{(\sin x + \cos x) - (\sin x - \cos x)}{\sin x + \cos x}dx\\ &= \frac{1}{2}\int_0^{2\pi} \left\{\int_0^{2\pi} dx + \int_0^{2\pi} \frac{(\sin x + \cos x)'}{\sin x + \cos x}dx\right\}\\ &= \frac{1}{2}\left(\frac{\pi}{2} + [\log|\sin x + \cos x|]_0^{\frac{\pi}{2}}\right)\\ &= \frac{\pi}{4} \quad ■\end{aligned}$$

88 ∗∗∗ $\displaystyle\int \frac{\sqrt{\tan x}}{\sin 2x}dx$

根号 → 基本的に外す

88

$$t = \sqrt{\tan x}$$
$$dt = \frac{1}{2\sqrt{\tan x}}\frac{1}{\cos^2 x}dx = \frac{1}{2t}\frac{1}{\cos^2 x}dx$$

$$\begin{aligned} I &= \int \frac{\sqrt{\tan x}}{\sin 2x}dx \\ &= \int \frac{t2t\cos^2 x dt}{2\sin x\cos x} \\ &= \int \frac{t^2}{\tan x}dt \\ &= t + C \\ &= \sqrt{\tan x} + C \quad ■ \end{aligned}$$

89 ∗∗∗ $\displaystyle\int \frac{\sin\frac{1}{x}}{x^3}dx$

三角関数の中身が複雑
→ 丸ごと置換

89

$$t = \frac{1}{x}$$
$$dt = -\frac{1}{x^2}dx$$
$$dx = -\frac{1}{t^2}dt$$

$$\begin{aligned} I &= \int \frac{\sin\frac{1}{x}}{x^3}dx \\ &= \int \frac{\sin t}{(\frac{1}{t})^3}(-\frac{1}{t^2})dt \\ &= \int t\sin t dt \\ &= t\cos t - \sin t + C \\ &= \frac{\cos\frac{1}{x}}{x} - \sin\frac{1}{x} + C \quad ■ \end{aligned}$$

90 ∗∗∗ $\displaystyle\int \frac{1}{\cos^4 x}dx$

三角関数は 2 乗に強い

90

$$\begin{aligned} \int \frac{1}{\cos^4 x}dx &= \int \frac{1}{\cos^2 x}\frac{1}{\cos^2 x}dx \\ &= \int (1+\tan^2 x)(\tan x)'dx \\ &= \int (\tan x)'dx + \int \tan^2 x(\tan x)'dx \\ &= \tan x + \frac{1}{3}\tan^3 x + C \quad ■ \end{aligned}$$

91 ∗∗∗ $\displaystyle\int \frac{1}{\sin x\cos x}dx$

$$(\log|\tan x|)' = \frac{1}{\sin x\cos x}$$

91

$$\int \frac{1}{\sin x\cos x}dx$$

$$(\log|\tan x|)' = \frac{(\tan x)'}{\tan x} = \frac{1}{\frac{\sin x}{\cos x}}\cdot\frac{1}{\cos^2 x} = \frac{1}{\sin x\cos x}$$

$$\int \frac{1}{\sin x\cos x}dx = \log|\tan x| + C \quad ■$$

92 ＊＊＊ $\displaystyle\int \frac{3\sin x - \sin 3x}{1+\cos x}dx$

-信州大学-
根号 → 基本的に外す

93 ＊＊＊ $\displaystyle\int \frac{1}{1+\sin x}dx$

$1-\sin x$ や $1-\cos x$ を見たら
a^2-b^2 の形に

94 ＊ $\displaystyle\int \frac{1}{\sin^2 x}dx$

–三角関数の積分–
すぐ積分できる形

1. $x^{\alpha} \xrightarrow[\text{積分}]{} \dfrac{1}{\alpha+1}x^{\alpha+1}$
2. $\dfrac{1}{x} \longrightarrow \log|x|$
3. $\cos x \longrightarrow \sin x$
4. $\sin x \longrightarrow -\cos x$
5. $a^x \longrightarrow \dfrac{1}{\log a}a^x$
6. $\dfrac{1}{\cos^2 x} \longrightarrow \tan x$
7. $\dfrac{1}{\sin^2 x} \longrightarrow -\dfrac{1}{\tan x}$

92
$$\begin{aligned}\int \frac{3\sin x - \sin 3x}{1+\cos x}dx &= \int \frac{3\sin x - (3\sin x - 4\sin^3 x)}{1+\cos x}dx\\ &= \int \frac{4\sin x(1-\cos^2 x)}{1+\cos x}dx\\ &= \int \frac{4\sin x(1-\cos x)(1+\cos x)}{1+\cos x}dx\\ &= \int 4\sin x - 2\sin 2x dx\\ &= -4\cos x + \cos 2x + C \quad ■\end{aligned}$$

93
$$\begin{aligned}\int \frac{1}{1+\sin x}dx &= \int \frac{1-\sin x}{1-\sin^2 x}dx\\ &= \int \frac{1}{\cos^2 x}dx - \int \frac{\sin x}{\cos^2 x}dx\\ &= \tan x - \frac{1}{\cos x} + C \quad ■\end{aligned}$$

94
$$\int \frac{1}{\sin^2 x}dx = -\frac{1}{\tan x} + C \quad ■$$

[証明]

$$\begin{aligned}\left(\frac{\cos x}{\sin x}\right)' &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x}\\ &= -\frac{1}{\sin^2 x}\end{aligned}$$

95 ＊＊＊＊＊ $\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{\tan x}dx$

根号 → 基本的に外す

95

$t=\sqrt{\tan x}$

$dx=\dfrac{2t}{1+t^4}dt$

x	0	$\to$	$\frac{\pi}{4}$
t	0	$\to$	1

$t-\dfrac{1}{\sqrt{2}}=\dfrac{\tan\theta}{\sqrt{2}}$

$dt=\dfrac{1}{\sqrt{2}}\dfrac{1}{\cos^2\theta}d\theta$

t	0	$\to$	1
t	$-\frac{\pi}{4}$	$\to$	α

$\tan\alpha=\sqrt{2}-1$

$\tan(\alpha+\beta)$

$=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$

$=\dfrac{2\sqrt{2}}{1-1}$

$=\infty$

$\alpha+\beta=\dfrac{\pi}{2}$

$$I=\int_0^{\frac{\pi}{4}}\sqrt{\tan x}dx$$
$$=\int_0^1\frac{2t^2}{1+t^4}dt$$
$$t^4+1=(t^2+\sqrt{2}t+1)(t^2-\sqrt{2}t+1)$$
$$I=\int_0^1\frac{2t^2}{(t^2+\sqrt{2}t+1)(t^2-\sqrt{2}t+1)}dt$$
$$I=\underbrace{\frac{1}{\sqrt{2}}\int_0^1\frac{t}{(t^2-\sqrt{2}t+1)}dt}_{I_1}\underbrace{-\frac{1}{\sqrt{2}}\int_0^1\frac{t}{(t^2+\sqrt{2}t+1)}dt}_{I_2}$$
$$I_1=\frac{1}{2}\int_0^1\frac{(t^2-\sqrt{2}t+1)'+\sqrt{2}}{t^2-\sqrt{2}t+1}$$
$$=\frac{1}{2}\int_0^1\frac{(t^2-\sqrt{2}t+1)'}{t^2-\sqrt{2}t+1}+\frac{\sqrt{2}}{(t-\frac{1}{\sqrt{2}})^2+\frac{1}{2}}dt$$
$$=\frac{1}{2}\left\{\Big[\log|t^2-\sqrt{2}+1|\Big]_0^1+2\int_{-\frac{\pi}{4}}^{\alpha}d\theta\right\}$$
$$=\frac{1}{2}\log\Big(2-\sqrt{2}\Big)+\alpha+\frac{\pi}{4}$$
$$I_2=\frac{1}{2}\log\Big(2-\sqrt{2}\Big)-\beta+\frac{\pi}{4}$$
$$I=\frac{1}{\sqrt{2}}\left(\frac{1}{2}\log\frac{2-\sqrt{2}}{2+\sqrt{2}}+\alpha+\beta\right)$$
$$=\frac{1}{\sqrt{2}}\left(\frac{1}{2}\log\Big(\sqrt{2}-1\Big)+\frac{\pi}{2}\right)\quad\blacksquare$$

96 ＊＊ $\displaystyle\int_0^{\frac{\pi}{6}}\cos^3 xdx$

微分系の接触を作る

96
$$\int_0^{\frac{\pi}{6}}\cos^3 xdx=\int_0^{\frac{\pi}{6}}(1-\sin^2 x)\cos xdx$$
$$=\int_0^{\frac{\pi}{6}}(\cos x-\sin^2 x\cos x)dx$$
$$=\left[\sin x-\frac{1}{3}\sin^3 x\right]_0^{\frac{\pi}{6}}=\frac{11}{24}\quad\blacksquare$$

97 ∗∗ $\displaystyle\int_0^{\frac{\pi}{2}} \sin^7 x dx$

微分系の接触を作る

98 ∗∗∗∗∗ $\displaystyle\int_0^{\frac{\pi}{4}} \frac{1}{\sin^2 x + 3\cos^2 x} dx$

-横浜国立大学-
$\tan x$ 置換の形

99 ∗∗∗∗∗ $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\log(\tan x)}{\cos^2 x} dx$

$\tan x$ 置換の形

97

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^7 x dx &= \int_0^{\frac{\pi}{2}} (1-\cos^2 x)^3 \cdot \sin x dx \\ &= \int_0^{\frac{\pi}{2}} \sin x - 3\cos^2 x \sin x \\ &\qquad + 3\cos^4 x \sin x - \cos^6 x \sin x dx \\ &= \left[-\cos x + \cos^3 x - \frac{3}{5}\cos^5 x + \frac{1}{7}\cos^7 x\right]_0^{\frac{\pi}{2}} \\ &= \frac{16}{35} \quad \blacksquare\end{aligned}$$

98

$t = \tan x$

$dt = \dfrac{1}{\cos^2 x} dx$

x	0	$\to$	$\frac{\pi}{4}$
t	0	$\to$	1

$t = \sqrt{3}\tan u$

$dt = \dfrac{\sqrt{3}}{\cos^2 u} du$

t	0	$\to$	1
u	0	$\to$	$\frac{\pi}{6}$

$$\begin{aligned}I &= \int_0^{\frac{\pi}{4}} \frac{1}{\sin^2 x + 3\cos^2 x} dx \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{\tan^2 x + 3} \cdot \frac{1}{\cos^2 x} dx \\ &= \int_0^1 \frac{1}{t^2+3} dt \\ &= \int_0^{\frac{\pi}{6}} \frac{1}{\frac{3}{\cos^3 u}} \frac{\sqrt{3}}{\cos^2 u} du \\ &= \left[\frac{\sqrt{3}}{3}\right]_0^{\frac{\pi}{6}} \\ &= \frac{\sqrt{3}}{18}\pi \quad \blacksquare\end{aligned}$$

99

$t = \tan x$

$dt = \dfrac{1}{\cos^2 x} dx$

x	$\frac{\pi}{4}$	$\to$	$\frac{\pi}{3}$
t	1	$\to$	$\sqrt{3}$

$$\begin{aligned}I &= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\log(\tan x)}{\cos^2 x} dx \\ &= \int_1^{\sqrt{3}} \log t dt \\ &= [t\log t - t]_1^{\sqrt{3}} \\ &= \frac{\sqrt{3}}{2}\log 3 - \sqrt{3} + 1 \quad \blacksquare\end{aligned}$$

100 *******

$$\int \frac{(\tan x+1)\sqrt{\sin^2 x e^{2\tan x}+\cos^2 x}}{\sin^3 x e^{2\tan x}}dx$$

tan x のたんは異端児の"たん"

100

$$t=\tan x$$
$$dt=\frac{1}{\cos^2 x}dx = (1+t^2)dx$$

$$u=te^t$$
$$du=(t+1)e^t dt$$

$$v=\sqrt{u^2+1}$$
$$vdv=udu = (1+t^2)dx$$

$$\begin{aligned}
I &= \int \frac{(\tan x+1)\sqrt{\sin^2 x e^{2\tan x}+\cos^2 x}}{\sin^3 x e^{2\tan x}}dx \\
&= \int \frac{(t+1)\sqrt{\frac{t^2}{t^2+1}e^{2t}+\frac{1}{t^2+1}}}{\frac{t^2}{t^2+1}\sqrt{\frac{t^2}{t^2+1}e^{2t}}}\frac{1}{t^2+1}dt \\
&= \int \frac{(t+1)\sqrt{t^2e^{2t}+1}}{t^3e^{2t}}dt \\
&= \int \frac{(t+1)e^t\sqrt{t^2e^{2t}+1}}{(te^t)^3}dt \\
&= \int \frac{\sqrt{u^2+1}}{u^3}du \\
&= \int \frac{u\sqrt{u^2+1}}{u^4}du \\
&= \int \frac{v^2}{(v^2-1)^2}dv \\
&= \frac{1}{4}\int \frac{v}{(v-1)^2}-\frac{v}{(v+1)^2}dv \\
&= \frac{1}{4}\int \frac{1}{v-1}+\frac{1}{(v-1)^2} \\
&\qquad -\frac{1}{v+1}+\frac{1}{(v+1)^2}dv \\
&= \frac{1}{4}\log|\frac{v-1}{v+1}|-\frac{v}{2(v^2-1)}+C \\
&= -\frac{\sqrt{u^2+1}}{21-(\sqrt{u^2+1})^2} \\
&\qquad +\frac{1}{4}\log\frac{\sqrt{u^2+1}-1}{\sqrt{u^2+1}+1}+C \\
&= -\frac{\sqrt{u^2+1}}{2u^2}+\frac{1}{2}\log|\frac{\sqrt{u^2+1}-1}{u}|+C \\
&= -\frac{\sqrt{t^2e^{2t}+1}}{2t^2e^{2t}} \\
&\qquad +\frac{1}{2}\log|\frac{\sqrt{t^2e^{2t}+1}-1}{te^t}|+C \\
&= -\frac{\sqrt{\tan^2 x e^{2\tan x}+1}}{2\tan^2 x e^{2\tan x}} \\
&\qquad +\frac{1}{2}\log|\frac{\sqrt{\tan^2 x e^{2\tan x}+1}-1}{2\tan x e^{\tan x}}|+C
\end{aligned}$$

■

0.4 積分公式

積分定数は省略する

1. $\displaystyle\int x^a dx = \frac{x^{a+1}}{a+1}\ (a \neq -1)$

2. $\displaystyle\int \frac{1}{x} dx = \log|x|$

3. $\displaystyle\int \sin x dx = -\cos x$

4. $\displaystyle\int \cos x dx = \sin x$

5. $\displaystyle\int \tan x dx = -\log|\cos x|$

6. $\displaystyle\int \tan^2 x dx = \tan x - x$

7. $\displaystyle\int \log x dx = x\log x - x$

8. $\displaystyle\int e^x dx = e^x$

9. $\displaystyle\int a^x dx = \frac{a^x}{\log a}$

10. $\displaystyle\int \frac{1}{\cos^2 x} dx = \tan x$

11. $\displaystyle\int \frac{1}{\sin^2 x} dx = -\frac{1}{\tan x}$

12. $\displaystyle\int \frac{1}{\sin x} dx = \frac{1}{2}\log\left(\frac{1-\cos x}{1+\cos x}\right)$

13. $\displaystyle\int \frac{1}{\cos x} dx = \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)$

14. $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n x dx = \begin{cases} \dfrac{(n-1)!!}{n!!} \\ \dfrac{\pi}{2}\dfrac{(n-1)!!}{n!!} \end{cases}$

15. $\displaystyle\int_a^b (x-a)^m (b-x)^n dx = \frac{m!n!}{(m+n+1)!}(b-a)^{m+n+1}$

16. $\displaystyle\int e^{ax}\cos bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\cos bx)$

17. $\displaystyle\int e^{ax}\sin bx dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx - b\cos bx)$

18. $\displaystyle\int \tan^n x dx$

$$= \begin{cases} \displaystyle\sum_{t=1}^{k}(-1)^{k-t}\frac{\tan^{2t} x}{2t} + (-1)^{k+1}\log|\cos x| & (n = 2k+1 \\ \displaystyle\sum_{t=1}^{k}(-1)^{k-t}\frac{\tan^{2t-1} x}{2t-1} + (-1)^{k} x & (n = 2k) \end{cases}$$

19. $\displaystyle\int (\log x)^n dx = x\sum_{k=0}^{n}(-1)^k {}_n\mathrm{P}_k (\log x)^{n-k}$

5, 6, 10 の証明 (異端児のたん)

$$\int \tan x dx = -\log|\cos x|$$

$$\int \tan^2 x dx = \tan x - x$$

$$\int \frac{1}{\cos^2 x} dx = \tan x$$

−5 の証明−

$$\begin{aligned} \int \tan x dx &= \int \frac{\sin x}{\cos x} dx \\ &= \int -\frac{(\cos x)'}{\cos x} dx \\ &= -\log|\cos x| + C \quad \blacksquare \end{aligned}$$

−10 の証明−

$$\begin{aligned} \int \frac{1}{\cos^2 x} dx &= \int (\tan x)' dx \\ &= \tan x + C \quad \blacksquare \end{aligned}$$

−6 の証明−

$$\begin{aligned} \int \tan^2 x dx &= \int \frac{1}{\cos^2 x} - 1 dx \\ &= \int \frac{1}{\cos^2 x} dx - \int dx \\ &= \tan x - x + C \quad \blacksquare \end{aligned}$$

12, 13 の証明

$$\int \frac{1}{\sin x}dx = \frac{1}{2}\log\left(\frac{1-\cos x}{1+\cos x}\right)$$
$$\int \frac{1}{\cos x}dx = \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)$$

–部分積分を使った解法–

$$\begin{aligned}\frac{1}{\sin x} &= \frac{\sin x}{\sin^2 x}\\ &= \frac{\sin x}{1-\cos^2 x}\\ &= \frac{\sin x}{(1+\cos x)(1-\cos x)}\\ &= \frac{1}{2}\left(\frac{\sin x}{1-\cos x}+\frac{\sin x}{1+\cos x}\right)\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sin x}dx &= \frac{1}{2}\int\left(\frac{\sin x}{1-\cos x}+\frac{\sin x}{1+\cos x}\right)dx\\ &= \frac{1}{2}\{\log(1-\cos x)-\log(1+\cos x)\}\\ &= \frac{1}{2}\log\left(\frac{1-\cos x}{1+\cos x}\right)+C \quad \blacksquare\end{aligned}$$

$$\begin{aligned}\frac{1}{\cos x} &= \frac{\cos x}{\cos^2 x}\\ &= \frac{\cos x}{1-\sin^2 x}\\ &= \frac{1}{2}\left(\frac{\cos x}{1+\sin x}+\frac{\cos x}{1-\sin x}\right)\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\cos x}dx &= \frac{1}{2}\int \frac{\cos x}{1-\sin x}+\frac{\cos x}{1+\sin x}dx\\ &= \frac{1}{2}\{\log(1+\sin x)-\log(1-\sin x)\}\\ &= \frac{1}{2}\log\left(\frac{1+\sin x}{1-\sin x}\right)+C \quad \blacksquare\end{aligned}$$

–ユニバーサル置換を用いる解法–

$\tan\dfrac{x}{2} = t$ とおく,

$$\begin{aligned}\sin x &= 2\sin\frac{x}{2}\cos\frac{x}{2}\\ &= 2\tan\frac{x}{2}\cos^2\frac{x}{2}\\ &= \frac{2t}{1+t^2}\\ \cos x &= 2\cos^2\frac{x}{2}-1\\ &= \frac{2}{1+t^2}-1 = \frac{1-t^2}{1+t^2}\\ \frac{dt}{dx} &= \frac{1}{2}\frac{1}{\cos^2\frac{x}{2}} = \frac{1}{2}(1+t^2)\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\sin x}dx &= \int \frac{1+t^2}{2t}\frac{2}{1+t^2}dt\\ &= \log|t|+C\\ &= \log|\tan\frac{x}{2}|+C \quad \blacksquare\end{aligned}$$

$$\begin{aligned}\int \frac{1}{\cos x}dx &= \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}dt\\ &= \int \frac{1}{1-t}+\frac{1}{1+t}dt\\ &= \log\left|\frac{1+\tan\frac{x}{2}}{1-\tan\frac{x}{2}}\right|+C \quad \blacksquare\end{aligned}$$

–微分形の接触を用いた解法–

$$\begin{aligned}\int \frac{1}{\sin x}dx &= \int \frac{dx}{2\sin\frac{x}{2}\cos\frac{x}{2}}\\ &= \int \frac{dx}{2\tan\frac{x}{2}\cos^2\frac{x}{2}}\\ &= \log|\tan\frac{x}{2}|+C \quad \blacksquare\end{aligned}$$

$\cos x = -\sin\left(x-\dfrac{\pi}{2}\right)$ に注意すると,

$$\begin{aligned}\int \frac{1}{\cos x}dx &= \int -\frac{1}{\sin\left(x-\frac{\pi}{2}\right)}dx\\ &= -\log|\tan\left(\frac{\pi}{2}-\frac{\pi}{4}\right)|+C \quad \blacksquare\end{aligned}$$

16, 17 の証明

$$\int e^{ax}\cos bx dx$$
$$= \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\cos bx)$$
$$\int e^{ax}\sin bx dx$$
$$= \frac{e^{ax}}{a^2+b^2}(a\cos bx - b\cos bx)$$

–複素指数関数を用いた証明–

オイラーの公式から

$$\int e^{(a+ib)x}dx = \int e^{ax}\cos bx + ie^{ax}\sin bx\ dx$$
$$\int e^{(a+ib)x}dx = \frac{e^{(ax+ibx)}}{a+bi} + C$$
$$= \frac{e^{ax}(\cos bx + i\sin bx)}{a+bi}\cdot\frac{a-bi}{a-bi}$$
$$= \frac{e^{ax}}{a^2+b^2}\{(a\cos bx + b\sin bx)$$
$$+ i(a\sin bx - b\cos bx)\}$$

実部と虚部をそれぞれ比較すると，公式が求まる．

n が奇数のとき

$$\int_0^{\frac{\pi}{2}}\sin^n x dx = \int_0^{\frac{\pi}{2}}\cos^n x dx = \frac{(n-1)!!}{n!!}$$

n が偶数のとき

$$\int_0^{\frac{\pi}{2}}\sin^n x dx = \int_0^{\frac{\pi}{2}}\cos^n x dx = \frac{\pi}{2}\frac{(n-1)!!}{n!!}$$

–部分積分と漸化式を用いた積分–

$$I_n = \int_0^{\frac{\pi}{2}}\sin x\sin^{n-1}x dx$$
$$= \left[-\cos x\sin^{n-1}x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x\cos^2 x dx$$
$$= (n-1)\int_0^{\frac{\pi}{2}}\sin^{n-2}x(1-\sin^2 x)dx$$
$$= -(n-1)I_n + (n-1)I_{n-2}$$

よって，以下の漸化式が成立する

$$I_n = \frac{n-1}{n}I_{n-2}$$

n が奇数のとき

$$I_n = \frac{n-1}{n}I_{n-2}$$
$$= \frac{n-1}{n}\frac{n-3}{n-2}I_{n-4}$$
$$= \cdots = \frac{(n-1)!!}{n!!}I_1 = \frac{(n-1)!!}{n!!} \quad \blacksquare$$

n が偶数のとき

$$I_n = \frac{(n-1)!!}{n!!}I_0$$
$$= \frac{\pi}{2}\frac{(n-1)!!}{n!!} \quad \blacksquare$$

また，$0 \to \frac{\pi}{2}$ の範囲のとき

$$\cos\left(\frac{\pi}{2}-x\right) = \sin x$$
$$\int_0^{\frac{\pi}{2}} f(\cos x)dxS = \int_0^{\frac{\pi}{2}} f(\sin x)dx$$

0.5 今後の展望

・jsbook にした際に生じた式の見切れを全部直したい (2023/07/03 修正)
・誤字脱字がどうしてもある
→ 一旦全部一通り解きたい (2023/07/03 修正)
・積分公式の証明は殆どは載せたい
・区分求積法を新しい項目として載せたい
→ 大学過去問からの引用を増やしたい
・海外の積分コンテストを漁って, より奇問を増やしたい (上級者向け)
・有理関数の積分がこんなに少ないはずがない

0.6 宣伝

このプリントに載っている積分問題をランダムに出題してくれる機構を作成しました. 難易度別の出題にも対応しています. ”がっくりの箱庭”で検索してみてください.

0.7 謝辞

塾の後輩である小野真士氏には, 本教材の創作にあたり, 全ての問題に実際に取り組み, 誤植修正してくださりました. また, Q73 の問題に至っては三角関数の合成を用いた新しい解法までご教授いただきました. ここに感謝の意を表します.

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